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A PRIOR AND DATA VALIDATION AND ADJUSTMENT SCHEME FOR BAYESIAN RELIABILITY ANALYSIS IN ENGINEERING DESIGN

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ABSTRACT

Bayesian reliability analysis (BRA) technique has been actively used in reliability assessment for engineered systems. However, there are two key controversies surrounding the BRA, that is, the reasonableness of the prior, and the consistency among all data sets. These issues have been debated in Bayesian analysis for many years, and as we observed, they have not been resolved satisfactorily. These controversies have seriously hindered the applications of BRA as a useful reliability analysis tool to support engineering design. In this paper, a Bayesian reliability analysis methodology with a prior and data validation and adjustment scheme (PDVAS) is developed to address these issues. In order to do that, a consistency measure is defined first that judges the level of consistency among all data sets including the prior. The consistency measure is then used to adjust either the prior or the data or both to the extent that the prior and the data are statistically consistent. This prior and data validation and adjustment scheme is developed for Binomial sampling with Beta prior, called Beta-Binomial Bayesian model. The properties of the scheme are presented and discussed. Various forms of the adjustment formulas are shown and a selection framework of a specific formula, based on engineering design and analysis knowledge, is established. Several illustrative examples are presented which show the reasonableness, effectiveness and usefulness of PDVAS. General discussion of the scheme is offered to enhance the Bayesian Reliability Analysis in engineering design for reliability assessment.

KEY WORDS

Reliability, Bayesian Reliability Analysis (BRA), Prior Distribution, Posterior Distribution, Prior and Data Validation and Adjustment Scheme (PDVAS)

1. INTRODUCTION

Bayesian reliability analysis (BRA) technique has been actively used in reliability assessment [1-4]. BRA follows a traditional Bayesian approach that is to assume a sampling distribution for the data being analyzed, assign a probability distribution, called prior distribution, to the sampling distribution parameters, then collect test data to form the likelihood function, and update the prior to the posterior distribution with the Bayes' formula that aggregates the prior with the likelihood function [3-4]. It has been long known that the reliability result obtained through BRA is very sensitive to the assignment of the prior. The criticisms on the prior assignment, and prior and data inconsistency have not been stopped since the Bayesian technique was introduced, as evidenced by many literature articles [5-6]. Numerous researches have been under way that addressed the initial prior assignment, and several objective and non-informative prior generation methods have been developed [7-8]. Examples of the approaches include Maximum Entropy method [9], Reference Analysis technique [10], and Frequentist Matching method [11-13].

After the initial prior is determined, the Bayesian then collects test or experiment or analysis data, and derives the posterior distribution using the Bayesian formula. This process is called Bayesian updating. The Bayesian updating can be conducted repeatedly as multiple data sets and new information become available. The Bayesian updating basically takes the prior and the experiment or test or analysis data, and aggregates them in a weighted average manner. All this is done procedurally, usually without considering how contradicting and inconsistent the prior is with the data as well as among data sets by Bayesian analysis itself.

Figure 1 describes the traditional Bayesian analysis flow. After a sampling distribution is specified in Step 1 and an initial prior is determined in Step 2, the Bayesian then collects experiment data to form likelihood function in step 3, and derives the posterior distribution in Step 4 using the Bayesian formula given by Eq. (1). If a repeated Bayesian updating is conducted, the posterior distribution derived from Step 4 loops back in Step 4R as a new prior input to Step 3, and the Bayesian formula (Eq. (1)) is used again to aggregates this new prior with new likelihood data to arrive an updated posterior distribution. The posterior distribution is then used as a statistical inference tool in engineering design applications in Step 5.

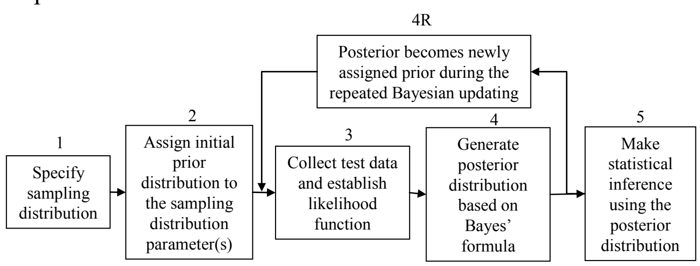

Figure 1. Traditional Bayesian Analysis Flow

$$f_{\Theta|X}(\theta \mid x) = \frac{f_{\Theta,X}(x,\theta)}{f_X(x)} = \frac{f_{X|\Theta}(x \mid \theta) f_\Theta(\theta)}{f_X(x)} \tag{1}$$

In Eq. (1), $f_\Theta(\theta)$ is the prior density for the sampling distribution parameter Θ; $f_{X|\Theta}(x \mid \theta)$ is the likelihood function of the data x given a Θ value as θ; $f_{\Theta|X}(\theta \mid x)$ is the posterior density of the Θ; $f_X(x)$ is the marginal density of the data.

If a prior distribution takes the same form of the distribution function as the posterior distribution, the prior is called conjugate prior. Mathematically, the prior, $f_\Theta(\theta)$ in a conjugate Bayesian model has the same function form as the posterior, $f_{\Theta|X}(\theta \mid x)$. The examples of conjugate prior include Beta prior with Binomial sampling, called Beta-Binomial Bayesian Model, and Normal prior for the mean with Normal sampling. For more conjugate prior examples, see [3-4]. A conjugate Bayesian model has several nice features. The first, the derivation of the posterior is much easier. Usually, a closed form formula exists that aggregates the prior and the data to obtain the posterior. For example, for Beta-Binomial model with Binomial parameter *p*, if we assume *p* is subject to $Beta(\alpha,\beta)$, and we obtain the test data of F failures out of N trials, the posterior distribution of *p* is then given by $Beta(\alpha+F,\beta+N-F)$. Secondly, for a repeated Bayesian updating with k (>1) data sets available, the final Bayesian updated result is independent of the sequence of the updating taken on the k data sets.

During engineering design, especially at conceptual design stage, reliability analysis relies on various data sources, including historical failure data from similar parts or systems, expert opinions, engineering modeling and simulations, and prototype or lab test results. Huang and Jin [14] provided a comprehensive survey of reliability prediction data sources as potential data inputs to BRA. Grantham Lough et al. [15-16] correlated the historical failure data to the functions of system during functional design to assist risk assessment. Wang and Jin [17] developed a functional design approach which utilized Bayesian Network technique with uniform distributions as inputs to assess individual function's influence on the system failure probability. Wang et al. [1] applied BRA technique during a product design and development cycle using evolving, insufficient and subjective data sets including customer survey, response surface physics model, and clinic trial test data for a repeated Bayesian updating as the design iterates. Huang and Jin [18] extended the traditional reliability stress-strength interference theory to the conceptual design with combination of team survey, historical similar function design data, physics bounds of the design to define conceptual stress and conceptual strength for reliability quantification. However, none of the above work has addressed the data inconsistency and data contradiction issue. For some BRA applications, an obvious data inconsistency may appear between the prior and the test data or among data sets from various data sources. One extreme example is that the prior states that the Binomial sampling parameter *p* is equal to 0.01 with probability 1 but the experiment data show 5 failures out of 10 trials which is very much contradicting with the prior from both statistical point view and common sense. A less extreme example is that the prior states that the Binomial parameter *p* is subject to the Beta(1, 100) prior while the experiment data shows 1 failure out of 10 trials. For the data samples like this, what Bayesian analysis produces is a weighted average of the prior and the data set as the posterior, as illustrated in Figure 2, though the likelihood data and the prior hardly overlap as shown in the figure. A data inconsistency example is that one data set has 1 failure and 2 successes and the other has 1 failure and 50 successes. It is not likely that these two data sets are from the same sampling distribution. Therefore it is seriously questionable whether these two data sets are combinable for Bayesian analysis. The aforementioned prior generation methods, namely, Maximum Entropy method, Reference Prior Analysis and Frequentist Matching only address the generation of the initial prior. As we surveyed Bayesian Analysis literature, we observe that there is little active research that addresses the prior and data inconsistency, and the validation of the prior and the data during a repeated Bayesian updating process. The research in [19] is probably one of the few that discuss the data conflict. This observation is confirmed by the statement in Wikipedia [http://en.wikipedia.org/wiki/Bayesian_probability]. "*Of the tens of thousands of papers published using Bayesian methods, few criticisms have been made of implausible priors in concrete applications.*"

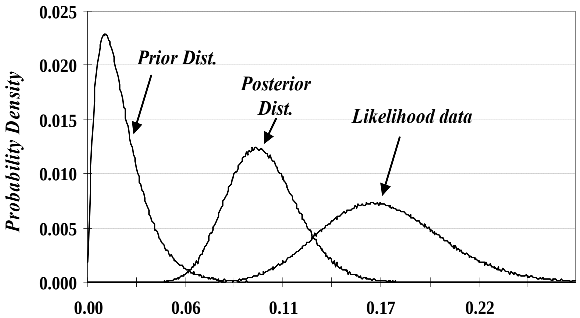

Figure 2. Bayesian Result as a Weighted Average of Prior and Data Set

The above discussion brings an obvious need for prior and data validation and adjustment. If the prior is contradicting with the data, the analyst has 3 choices: 1) accept the prior and the data, and perform Bayesian update as usual; 2) reject the prior and the data; or 3) do something about the prior and the data to continue the Bayesian process in a reasonable manner. For Choice 1, the analyst may lead himself or herself to a misleading inference model generated from Bayesian. For Choice 2, the analyst will have no data to perform Bayesian analysis. For Choice 3, the literature survey indicates there is no existing theory and method doing so. The objective of this paper is to provide such a method helping address Choice 3. In Section 2, we present a modified Bayesian updating process with an added step that validates the consistency between prior and data, and adjusts the prior and the data accordingly if inconsistencies arise. We provide the mathematical formulas for the prior and data validation and adjustment scheme (PDVAS) for Beta-Binomial Bayesian model. In Section 3, we present and discuss several properties of PDVAS which provide some insight and motivation of PDVAS. In Section 4, we discuss the selection of prior and data adjustment formulas based on the engineering design knowledge and data available. In Section 5, we present various examples to illustrate the PDVAS applications. We then summarize the paper in Section 6, and discuss future research possibilities including the generalization of PDVAS to other prior-data sampling model situations.

2. A Prior and Data Validation and Adjustment Scheme (PDVAS)

2.1 Modified Bayesian Analysis Flow

Figure 3 presents our proposal of the modified Bayesian analysis process with an added prior and data validation and adjustment step, which is Step 4a. All other steps are the same as the original Bayesian analysis flow as in Figure 1. We discuss the details of the Step 4a in the next several sections.

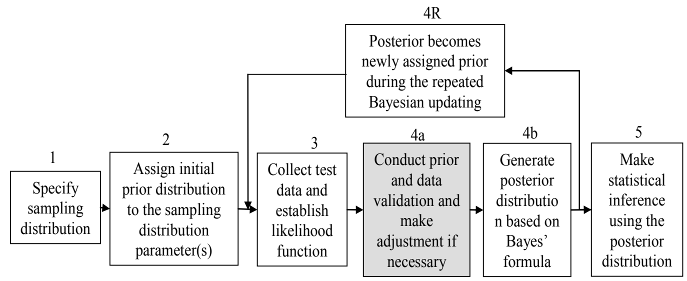

Figure 3. Altered Bayesian Analysis Flow with Addition of Prior and Data Validation and Adjustment Step

2.2 Prior and Data Validation and Adjustment Formulas

For Step 4a of Figure 3 for a Beta-Binomial Model, we have a Binomial distribution as the sampling distribution with the parameter p as follows,

$$p(X = x \mid \theta = p) = \binom{N}{x} p^x (1-p)^{(N-x)} \tag{2}$$

x = 0, 1, 2,…, N. p is assigned a Beta distribution with Beta parameters α and β. Its density is

$$f(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{Beta(\alpha,\beta)} \tag{3}$$

In Eq. (3), $0 < p < 1, \alpha > 0$ *and* $\beta > 0$. $Beta(\alpha,\beta)$ is a complete Beta function given by $\int_0^1 p^{\alpha-1}(1-p)^{\beta-1}dp$. The mean and the standard deviation of the Binomial sampling distribution are

$$\mu_{Bino} = \mathrm{N}p \tag{4}$$

and $$\sqrt{Np(1-p)} \tag{5}$$

respectively.

The mean and the standard deviation of the Beta(α, β) are

$$\mu_{Beta} = \frac{\alpha}{\alpha+\beta} \tag{6}$$

and $$\sigma_{Beta} = \frac{\sqrt{\alpha\beta}}{(\alpha+\beta)\sqrt{\alpha+\beta+1}} \tag{7}$$

respectively.

For a general Bayesian analysis, we have the following data: Prior distribution is given by Beta(α, β), and k data sets (k≥1) are given by (F_1, S_1), (F_2, S_2),…, (F_k, S_k). For the convenience of the notation, we just name $F_0 \equiv \alpha$ and $S_0 \equiv \beta$. So we have Bayesian data sets: (F_0, S_0), (F_1, S_1), (F_2, S_2),…, (F_k, S_k). Recall F_i and S_i represent number of failures and number of successes in the i-th data set respectively. With the traditional Bayesian process, we obtain the Beta posterior, Named Beta(α^*, β^*). Using Eq (1), we get

$$\alpha^* = \sum_{i=0}^{i=k} F_i \tag{8}$$

and $$\beta^* = \sum_{i=0}^{i=k} S_i \tag{9}$$

Remember that α^* and β^* are calculated without evaluating the data inconsistency and contradictions among (F_0, S_0), (F_1, S_1), (F_2, S_2),…, (F_k, S_k). Now the question is how we assess the data consistency and provide a measure of it. We define a consistency statistic with a probability associated with a χ^2 *statistic* as follows

$$\chi_C^2 = [\sum_{i=0}^{i=k}(F_i + S_i)]\left(\frac{\sum_{i=0}^{i=k}\frac{F_i^2}{F_i + S_i}}{\sum_{j=0}^{j=k}F_j} + \frac{\sum_{i=0}^{i=k}\frac{S_i^2}{F_i + S_i}}{\sum_{j=0}^{j=k}S_j} - 1\right) \tag{10}$$

The χ_C^2 formula is originated from the χ^2 statistic of the traditional hypothesis testing for 2 x (k+1) contingency table [20-21]. χ_C^2 has a degree of freedom k. Therefore, the consistency for the data sets (F_0, S_0), (F_1, S_1), (F_2, S_2),…, (F_k, S_k) is defined as

$$\text{C} \equiv \textit{Consistency} = P(\chi^2 > \chi_C^2) \tag{11}$$

Where χ_C^2 in Eq. (10) is a χ^2 random variable with degree of freedom k.

The motivation of the consistency formulas defined by Eq. (10) and (11) is that F_0+S_0 ($\equiv \alpha + \beta$) of the prior, per the Bayesian process, represents the prior's sample size, embedded in the prior knowledge in processing the posterior distribution [4]. F_0 ($\equiv \alpha$) and S_0 ($\equiv \beta$) approximately represent the number of failures and successes respectively, afforded by the prior. The mean of the prior is $\frac{F_0}{F_0 + S_0}$. If the data are consistent, all the data means, namely, $\frac{F_1}{F_1 + S_1}$, $\frac{F_2}{F_2 + S_2}$,…, $\frac{F_k}{F_k + S_k}$ should not be very far away from $\frac{F_0}{F_0 + S_0}$. As matter of fact, the data sets of (F_1, S_1), (F_2, S_2),…, (F_k, S_k) , according to the Bayesian assumption, all should be generated from the Binomial sampling with the parameter p which is subject to a Beta Prior with mean $\frac{F_0}{F_0 + S_0}$. Therefore, when the failure fraction, $\frac{F_i}{F_i + S_i}$, i=1, 2, …, k, of the data sets are equal to or close to the failure fraction of the prior, $\frac{F_0}{F_0 + S_0}$, we, to the maximum extent, believe that the data are consistent, and there is no contradiction between the prior and the data sets. Conversely, if one or more of the $\frac{F_i}{F_i + S_i}$ are drastically different from $\frac{F_0}{F_0 + S_0}$, or some $\frac{F_i}{F_i + S_i}$ is drastically different from $\frac{F_j}{F_j + S_j}$ $(i \neq j)$, we have a reason to think that the data are not consistent and the assumption that all data sets (F_1, S_1), (F_2, S_2),…, (F_k, S_k) are generated from the same Binomial sampling with p as the parameter is not adequate. Mathematically, when $\frac{F_0}{F_0 + S_0} = \frac{F_1}{F_1 + S_1} = ... = \frac{F_k}{F_k + S_k}$, Eq.(10) leads to χ_C^2 = 0. Therefore, the consistency C by Eq.(11) = 1. When some $\frac{F_i}{F_i + S_i}$ is very much different from some $\frac{F_j}{F_j + S_j}$ $(i \neq j)$, χ_C^2 in Eq. (10) becomes very big. Therefore, the consistency C by Eq.(11) goes to zero.

The consistency measure C, calculated by Eq. (10) and (11), represents the prior and data validation result. C is a value between 0 and 1. When C = 1 or close to 1, we believe that the data sets, (F_0, S_0), (F_1, S_1), (F_2, S_2),…, (F_k, S_k), are completely consistent or nearly consistent. Thereby we will implement the Bayesian updating unconditionally as we do in the traditional Bayesian process. When C = 0 or close to 0, we believe that the data sets, (F_0, S_0), (F_1, S_1), (F_2, S_2),…, (F_k, S_k), are completely inconsistent or nearly inconsistent. Thereby we will seriously challenge the assumption embedded in Bayesian that all data sets, (F_1, S_1), (F_2, S_2),…, (F_k, S_k), are generated from the same Binomial sampling with p as the parameter.

Now the question is that *what do we do when the consistency C by Eq. (11) is between 0 and 1?* We propose the following data adjustment algorithm as part of the Bayesian updating. This is the substantiation of Step 4a in our proposal of the altered Bayesian updating process presented in Figure 3. Figure 4 shows the algorithm and Figure 5 presents the data adjustment step details. Where $\chi_{C,m}^2$ in Figure 5 is the χ^2 statistic by Eq. (10) using data sets (F_0, S_0), (F_1, S_1),…, (F_i, S_i), but excluding the data set (F_m, S_m). Therefore $\chi_{C,m}^2$ has a degree of freedom of i-1. C_m is the consistency measure among data sets (F_0, S_0), (F_1, S_1),…, (F_i, S_i), excluding (F_m, S_m), therefore, $C_m = P(\chi^2 > \chi_{C,m}^2)$. The data adjustment step (Step 3) in Figure 4 and the adjustment details (Steps 3.1 – 3.3) in Figure 5 basically detect the inconsistency sources and adjust the data sets to make them consistent. The targeted data set ((F_v, S_v) in Step 3.2 and 3.3) for the data adjustment is the one with the biggest consistency value after it is excluded from the consistency calculation, therefore it is identified as the source of inconsistency. It is noted in Figure 4 that the data validation and adjustment starts from the second Bayesian updating since for the first update, we only have the initial prior and the first data set which won't provide us any direction how we can adjust the data. Step 2 and Step 3 in Figure 4 are in an iteration loop. Therefore there is a convergence issue. We will discuss this in next section. Step 3.3 is to adjust the data set (F_v, S_v), when it is found that it is the source of the inconsistency.

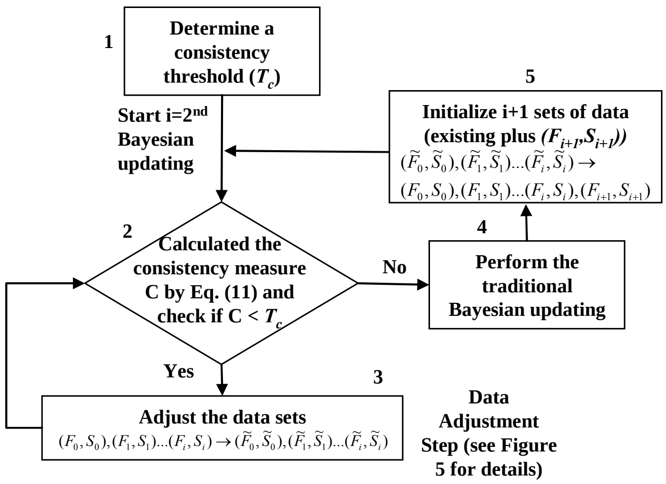

Figure 4. Bayesian Updating Prior and Data Adjustment Algorithm

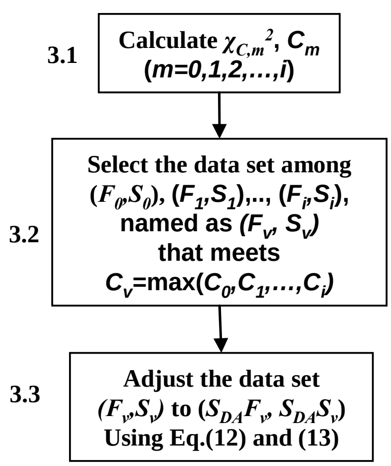

Figure 5. Details of the Data Adjustment Step

Next we discuss the data adjustment formula. We first define a data adjustment as a mapping from the consistency measure to the data adjustment score, denoted as S_{DA}.

$$S_{DA} = f(\text{Consistency}) \quad (12)$$

The exact form of f function in Eq. (12) will be determined in Section 4. The S_{DA} also takes values in the range of [0, 1]. We then apply the S_{DA} to the data set (F_v, S_v) in the following discounting manner to obtain the discounted values of F_v and S_v, namely, $\tilde{F}_v$ and $\tilde{S}_v$, respectively.

$$\tilde{F}_v = S_{DA}F_v \text{ and } \tilde{S}_v = S_{DA}S_v \quad (13)$$

So far, we haven't defined the detailed f functional forms of S_{DA} yet. But we know S_{DA} should satisfy the following conditions

$$S_{DA}(0) = f(0) = 0 \quad (14)$$

and $$S_{DA}(1) = f(1) = 1 \quad (15)$$

and $S_{DA}(\text{consistency}_2) \geq S_{DA}(\text{consistency}_1)$ when

$$\text{Consistency}_2 > \text{Consistency}_1 \text{ (monotone increase)} \quad (16)$$

Eq. (13), (14), (15) and (16) basically state that when the consistency measure is zero, we completely ignore the data set (F_v, S_v). When the consistency measure is one, we completely accept the data set (F_v, S_v) which is what the traditional Bayesian updating process does. The bigger the consistency measure, the less we discount the data. Recall that Step 1 of Figure 4 asks for defining a threshold value of the consistency measure (T_c). The role and the interpretation of this threshold are similar to the concept of the significance level in a traditional statistical hypothesis testing. But the rejection criterion in hypothesis testing is a step function that is when the probability value (*p-value*) of observing certain data sample is below the significance level, the null hypotheses will be rejected. In the context of the PDVAS application, we extend this step function to a set of smooth curves, which can incorporate engineering design and analysis knowledge for data adjustment. In this section, we present a general forms of S_{DA} Curves first. In next section, we will provide recommendations how a specific curve can be selected based on available engineering design and analysis data.

Figure 6 depicts various possibilities of S_{DA} curves with T_c = 0.05. For all the curves in the figure, S_{DA} = 1 when the consistency measure ≥ 0.05. The curve A in the figure is very close to taking all S_{DA} values of 1 for any consistency measure. So it is, if adopted, leading to the traditional Bayesian updating process. The curve G takes almost all S_{DA} values of 0 for the consistency measures ≤ 0.05, which leads to the nearly complete rejection of the data set (F_v,S_v), similar to the situation of a traditional hypothesis testing with a significance level of 5% of rejecting a null hypothesis. Section 4 will discuss the details how to select a T_c and S_{DA} curve.

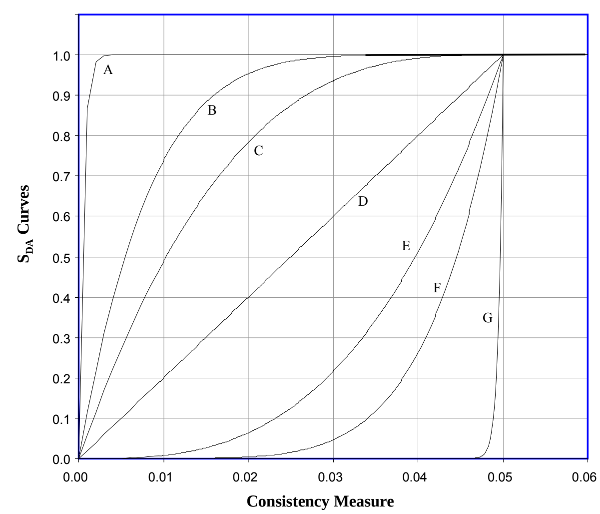

Figure 6. Potential Candidate Functions for S_{DA}

3. Discussion of Properties of PDVAS

Proposition 1. For the data set (F_v, S_v) being adjusted in Eq. (13), PDVAS does not change the mean of the data set but increases its standard deviation σ.

The proof of Proposition 1 is given in Appendix A. Proposition 1 reflects a key PDVAS strategy, that is to maintain the mean of the adjusted data set but increase the spread of the data distribution. In another words, PDVAS takes the face value embedded in the data regarding to the knowledge of central tendency but discounts the value of the data set by increasing its variance. The rationale of doing so is that when the prior or the data are collected, usually the uncertainty associated with the variability is much bigger than the one associated with the central tendency. Therefore the variability of the data is more doubtful than the mean. PDVAS focuses on the adjustment of the variability to achieve the data consistency.

Proposition 2. At the i-th Bayesian updating with the data sets (F_0,S_0), (F_1,S_1), ..,, (F_i,S_i), If

$$\frac{F_v}{F_v+S_v} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$, where $0 \le v \le i$, we then have

$$\frac{F_v}{F_v+S_v} < \frac{F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{F_v+S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} \le \frac{\widetilde{F}_v + \sum_{j=0,i\neq v}^{j=i} F_j}{\widetilde{F}_v+\widetilde{S}_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$

$$\equiv \frac{S_{DA}F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} \qquad (17)$$

Note if $\frac{F_v}{F_v+S_v} > \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$, Eq. (17) still holds but with all inequality signs reversed.

The proof of Proposition 2 is given in Appendix A. Proposition 2 indicates that when PDVAS detects a data inconsistency with the data set (F_v,S_v) as the source of the inconsistency, it adjusts (F_v,S_v) to reduce its weight in the posterior distribution such that the posterior mean is moving away from the mean of the data set (F_v,S_v) toward the mean of $(\sum_{j=0,j\neq v}^{j=i} F_j, \sum_{j=0,j\neq v}^{j=i} S_j)$. Under the extreme case, that is S_{DA} is zero or close to zero, (F_v,S_v) is completely or nearly completely ignored, and the posterior mean is the mean of $(\sum_{j=0,j\neq v}^{j=i} F_j, \sum_{j=0,j\neq v}^{j=i} S_j)$.

Proposition 3. For the data sets (F_0,S_0), (F_1,S_1), ..,, (F_i,S_i), there always exists a set of S_{DA} values such that Step 2 and 3 of Figure 4 will converge.

The proof of Proposition 3 is given in Appendix A. Proposition 3 confirms that there is always a data adjustment solution for PDVAS when a data inconsistency is detected, and that Step 2 and 3 of Figure 4 will not fall into a dead loop. In reality, Step 2 and 3 often take as a few as one or two iterations as the illustrative examples will show in Section 5.

Proposition 4. For no failure situation, that is, in (F_0,S_0), (F_1,S_1), ..,, $(F_k S_k)$, all F_i=0 except F_0, χ_C^2 by Eq. (10), after substituting F_0 by $\widetilde{F}_0 = S_{DA}F_0$ and S_0 by $\widetilde{S}_0 = S_{DA}S_0$, is a monotone increase function of S_{DA}, $\chi_C^2 \ge \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k} S_i$ and

$\chi_C^2 \to \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k} S_i$ when $S_{DA} \to 0$.

The proof of Proposition 4 is given in Appendix A. Proposition 4 addresses a special data situation, that is, there is no failure in the data. Therefore, there is no evidence of data inconsistency among (F_1,S_1), ..,, $(F_k S_k)$, since all data means = $F_j/(F_j+S_j)$=0 ($j=1,2,\ldots,k$). For the same reason, the failure ratio of the data can not be compared with the prior's. If we attempt to adjust (F_0,S_0), the proposition states that the adjusted χ^2 *value,* χ_C^2, has a lower bound limit. Therefore the data adjustment won't make χ_C^2 go to zero even S_{DA}=0. Thereby the prior and data validation and adjustment can not be performed meaningfully. PDVAS will not make any data adjustment under "no-failure" situation. PDVAS will default to the traditional Bayesian result under no-failure situation.

Proposition 5. The final posterior in a repeated Bayesian updating with multi-data sets using PDVAS is update sequence dependent.

Proposition 5 indicates that PDVAS eliminates the nice feature of the updating sequence independence possessed by the conjugate prior Bayesian process. To prove the proposition, we only need to provide an example illustrating the posterior distribution will vary from different updating sequences in a multi-data set Bayesian updating process using PDVAS. This example will be given in Section 5 (Example 4). We will also show in Section 5 through simulated examples, that stochastically, PDVAS is unbiased with different updating sequences. That is

$$E(\alpha_{k,sequence\_1}) = E(\alpha_{k,sequence\_2}) \text{ and}$$
$$E(\beta_{k,sequence\_1}) = E(\beta_{k,sequence\_2}) \qquad (18)$$

Here, E represents the expectations; $\alpha_{k,sequence\_1}$ and $\beta_{k,sequence\_1}$ are the *Beta* parameters of the posterior through k updating times for the updating sequence 1, and $\alpha_{k,sequence\_2}$ and $\beta_{k,sequence\_2}$ are the *Beta* parameters through a different k updating sequence. We will also show through simulations in Section 5 that the mean of the PDVAS posterior converges to the true data mean.

4. Selection of Data Adjustment Formulas

In Section 2, we discussed some general forms of the formulas being applied to data adjustment after PDVAS detects a data inconsistency. Proposition 3 in Section 3 indicates that we can always find a data adjustment score value, S_{DA} that discounts the data to make the data consistent. However, the intention of PDVAS is to discount the data as less as possible, especially be careful not throwing out good data. Therefore, the selection of S_{DA} value and its function is essentially important. If S_{DA} is too small (close to 0), we tend to disregard good data which defeats the original intention of the Bayesian analysis. If S_{DA} is too big (close to 1), we are accepting the data blindly which may ignore possible data inconsistency and lead to misleading inference. In this section, we first provide a family of S_{DA} functions then recommend some criteria for selecting a specific one for general engineering design applications.

4.1 A Family of S_{DA} curves

We select the incomplete Beta function within the range $[0, T_c]$ to represent the S_{DA} functions. Here T_c is the threshold value defined in Step 1 of Figure 4. When the consistency measure C, calculated by Eq. (11), meets $C \geq T_c$, we accept all the data as-is without any adjustment. When the C values fall within $(0, T_c)$, we calculate S_{DA} using an incomplete Beta function as follows.

$$S_{DA} = f(C) = \begin{cases} 0 & \text{if } C \leq 0 \\ \dfrac{1}{T_c}\displaystyle\int_0^{\frac{C}{T_c}} \dfrac{(\frac{u}{T_c})^{m-1}(1-\frac{u}{T_c})^{n-1}}{Beta\,(m,n)}du & \text{if } 0 < C < T_c \\ 1 & \text{if } T_c \leq C \end{cases} \qquad (19)$$

Beta(m,n) in Eq. (19) is a complete Beta function given by $\int_0^1 u^{m-1}(1-u)^{n-1}du$. The rationale of selecting the incomplete Beta function for S_{DA} is its versatility in its shapes and the easy interpretation of the parameters m and n in Eq.(19) related to the engineering design knowledge. To illustrate the merit of the Beta function, we present an example of a set of Beta curves in Figure 7 with $T_c = 0.05$. All the curves have the saddle points (the second derivative=0) near $m/(m+n)0.05$=0.0125 on the x-axis. As m and n get bigger, the curves get steeper around the saddle point, approaching the situation either PDVAS completely rejects the data or completely accepts the data depending on whether C is bigger than or smaller than 0.0125. It is recognized from probability theory that the consistency measure C by itself is a random variable with Uniform(0,1) as its distribution, since it is a value on the accumulative probability function of the χ^2 random variable by Eq. (11). Therefore, when C falls within (0, 0.05), it uniformly takes a value between 0 and 0.05. The family of the curves in Figure 7 basically states that there is on the average 1 out of 4 or 25% (=0.0125/0.05) chance to significantly reject the data. At the extreme case that we are certain 1 out of 4 times that the data are bad, we completely reject that data (S_{DA} = 0 or close to 0). However, in reality we make the statement "1 out of 4" with uncertainty, so we only reject data partially which is quantified by the C value with various curves. *Now the question is how we select T_c, m and n for the S_{DA} function for our PDVAS implementation?*

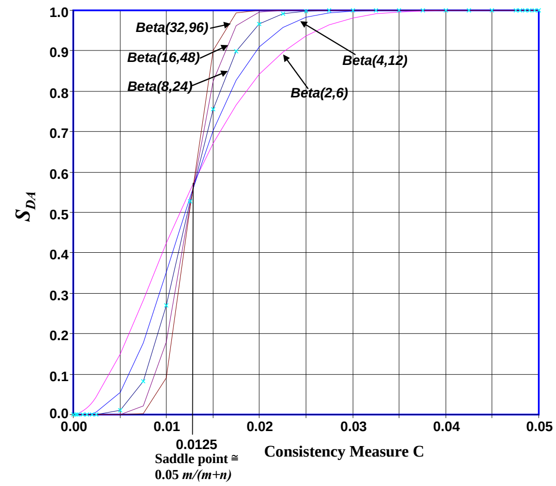

Figure 7. A Family of Beta Curves as Potential S_{DA} Functions

4.2 Selection of T_c, m and n

4.2.1 T_c selection

T_c represents a screening criterion which is conceptually similar to the significance level or α value in a traditional statistical hypothesis testing. The α value is always set to be small (≤0.1) to avoid unacceptable false positive in the hypothesis testing. From engineering design point view, when we use Bayesian analysis to aggregate the data to predict reliability, data often come from various sources therefore big uncertainty can be expected. While we want data inconsistency to be detected, we don't want to go through the data adjustment step when the evidence of data inconsistency is not strong. Therefore, we recommend using T_c value of 0.05 as a standard value for the inconsistency screening. For the case that we want to closely mimic traditional Bayesian analysis result without concerning too much about data inconsistency, we can use a T_c value of 0.01 or smaller.

4.2.2 m,n selection

As we mentioned in Section 4.1 that the ratio of *m/(m+n)* approximately represents our knowledge and judgment based on engineering design and analysis data, on the possible percentage of inconsistent data. While keeping *m/(m+n)* unchanged, the more certain we can pinpoint the source of data inconsistency, the bigger the m and n we can assign to. As one of our early research papers surveyed [14], reliability data for a Bayesian Analysis can come from the following 4 sources: 1) Statistical frequency method (SF); 2) Similarity and Comparative Assessment (SCA); 3) Physics Based Modeling and Simulation (PBMS); and 4) Expert Elicitation (EE). Usually, SF data, which are generated from field or lab simulated operating environment, have the smallest modeling uncertainty for the reason that fitted statistical inference model partially addresses the sampling modeling uncertainty. Uncertainty in other three data sources (SCA, PBMS and EE) can vary widely but data from EE probably have the biggest modeling uncertainty since they are purely based on expert judgment. With the above assessment, we use Table 1 as the selection frame work for *m* and *n* which also serves as an illustrative example.

Col (1) in Table 1 classifies the data source categories. We use our research result in [14] to divide all possible data sources into 4 categories (SF, SCA, PBMS and EE). Users of PDVAS can create their own data source categories. Col (2) is the engineering judgment of the analyst based on their knowledge on the design and analysis for what the average percentage of the inconsistent data can be. Col (3) is to assess the uncertainty of Col (2) which asks approximately how many times the analyst has experienced the data inconsistency instances in the past. Col (4) just takes Col (3) as *m*. Col (5) is to back-calculate *n* using Eq.(20). Col (6) completely defines the S_{DA} *Beta* function used in Eq.(19). Figure 8 creates the S_{DA} Beta curves based on the data from Table 1.

$$\frac{m}{m+n} = \text{Col (2)} \tag{20}$$

Figure 8 indicates that we rarely reject SF data in a dramatic manner shown in Curve (1). This is in line with our recognition that SF data often is the most reliable data source. Comparing Curve (2) (*Beta (2,2)*) and Curve 3 (*Beta(1,1)*), both of them partially discount the data in a prorated fashion. Curve (3) is strictly liner. Curve (2) discounts the data less when C > 0.025 and discounts more when C < 0.025 than Curve (3) does. This is because we are more confident on Beta(2,2) than on Beta(1,1), which by derivation, indicates we have experienced more bad data instances in Beta(2,2) situation than in Beta(1,1). For Curve (4) in the figure, we discount the majority of the data (more than 50%) when C falls below 0.035 (70% of 0.05).

Understanding the assessment on the data merits based on the data categories is subject to debate and assignment of *m* and *n* are subjective, the above approach is only considered to be a framework. Users of PDVAS can create their own data source categories, and assign *m* and *n* with their own knowledge and judgment. We recommend, for a quick analysis or for the situations that not much information is available about the data sources, the user uses the linear curve (*Beta(1,1)*) as the S_{DA} function as Figure 9 shows, which basically discounts the data in the linearly pro-rated manner when $C \le T_c$ without assessing the fraction of possible bad data. In many of our PDVAS simulation runs, this approach is proven to be reasonable.

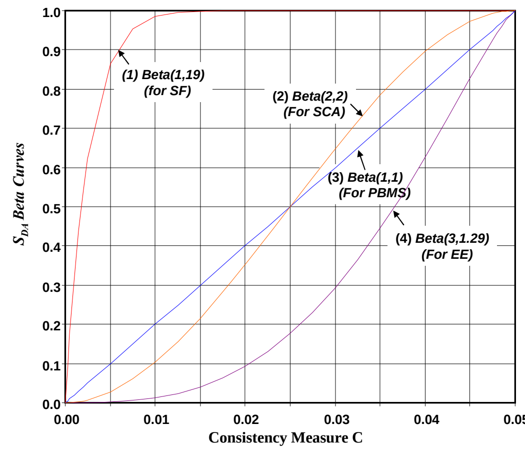

Figure 8. S_{DA} Curves for the Data in Table 1

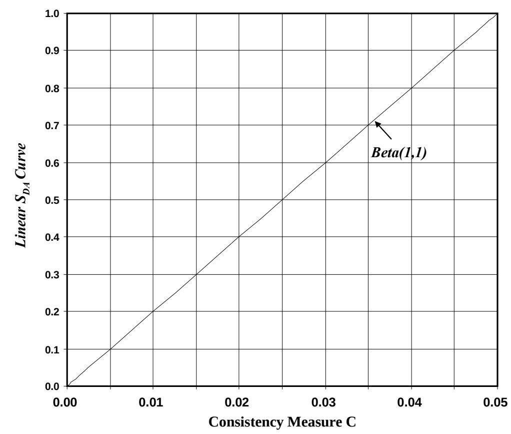

Figure 9. Linear S_{DA} curve

5. Illustrative Examples of PDVAS Applications

In this section, various examples are presented to illustrate PDVAS applications. The results from Example 1 to 4 are obtained through Monte Carlo simulations that assume certain Bayesian prior and data sampling distributions. Example 1 to 3 show the effectiveness of PDVAS that detects data inconsistencies and adjusts the data adequately. Example 4 shows the updating sequence dependency of PDVAS. Example 5 is to apply PDVAS to a rocket engine reliability analysis with various data sources.

Example 1. We assume the initial prior distribution is Beta(1,99) but the sampling distributions all come from Binomial with p = 0.10. Intuitively, there is a data inconsistency between the prior and the data sets because the mean of prior is 1/(1+99)=0.01 while sampling p=0.10. We run 20 Bayesian updates following the process outlined in Figure 3. The results are obtained through a computer simulation using the PDVAS algorithm presented in Figure 4 and 5. We use T_c = 0.05 and the linear curve for S_{DA} as in Figure 9 to execute PDVAS. Figure 10 presents the mean comparison. Line (1) in the figure is the true sampling mean (*p*=0.1). Curve (2) is the Monte Carlo simulated average means from 10,000 Monte Carlo runs using PDVAS. Curve (3) is the average means of

Table 1. *m* and *n* Selection Framework with an Example

(1)	(2)	(3)	(4)	(5)	(6)
Data category	Estimated % of inconsistent data	Number of Analyst's experiences on bad data	m	n	S_{DA} *Beta* Function in Eq. (19)
Statistical Frequency (SF)	5%	1	1	19	Beta(1,19)
Similarity and Comparative Assessment (SCA)	50%	2	2	2	Beta(2,2)
Physics Based Modeling and Simulation (PBMS)	50%	1	1	1	Beta(1,1)
Expert Elicitation (EE)	70%	3	3	1.29	Beta(3,1.29)

traditional Bayesian updated posteriors. This example illustrates that PDVAS is effective in correcting the data for the case initial prior is too optimistic. The figure also indicates that all three curves tend to converge together eventually when *i* (number of updates) goes to infinite. However, PDVAS mean is much closer to the true sampling mean for the small number of updates. Therefore, PDVAS is very useful for the practical situations with small number of Bayesian updates. Figure 11 presents the variance comparisons, which also indicates that PDVAS predicts the posterior variance much closer to the true sampling variance than the traditional Bayesian does. This is because the traditional Bayesian blindly takes the initial prior *Beta(1,99)* as part of the posterior updating which significantly increases total sample size in the final posterior distribution, therefore under-estimates the variance.

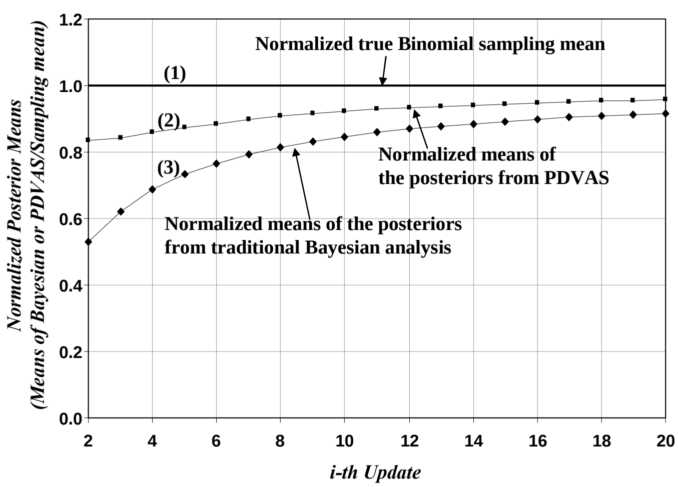

Figure 10. Mean Comparisons of PDVAS and Traditional Bayesian for Example 1

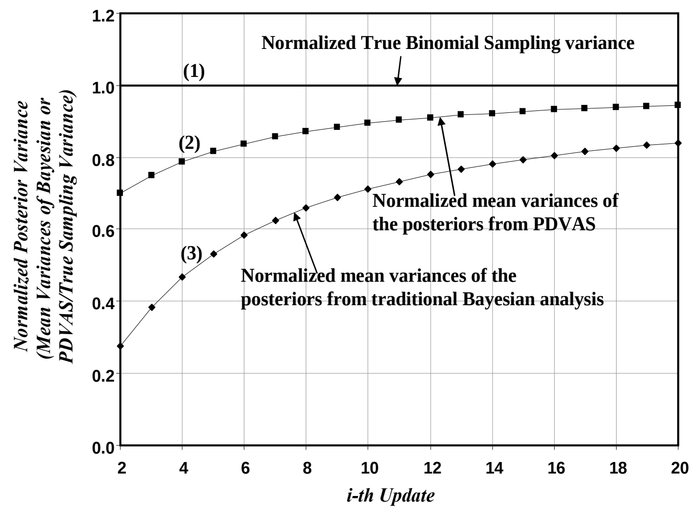

Figure 11. Variance Comparisons of PDVAS and Traditional Bayesian for Example 1

Example 2. Similar to Example 1, we assume the sampling distribution is a Binomial with p = 0.10. Initial prior is the non-informative prior *Beta(0.5, 0.5)*. However, in this example, we insert a data inconsistency anomaly by randomly generating 10% of the data sampling with Beta(3,97) as the sampling distribution. Notice Beta(3,97) has a much smaller mean than the Binomial Sampling's with p = 0.10. This is, in a simulated practical application, to test whether PDVAS can detect the data inconsistency as a multiple Bayesian updating is executed to incorporate newly obtained data. Again we use T_c = 0.05 and the linear S_{DA} curve. Figure 12 and 13 present the mean and variance comparisons respectively. The Results from the figures show PDVAS is superior to the tradition Bayesian since both means and variances of PDVAS posteriors are closer to the true mean and variance than the tradition Bayesian posteriors. It is interesting to notice that both PDVAS and traditional Bayesian means diverge from the true sampling mean. This is because there is always 10% of the sampling data with *Beta(3,97)* as their distribution with a significant smaller mean (3/(3+97) = 0.03) than the sampling mean (0.10). PDVAS for this case is not detecting all data inconsistency which is intentional in the design of PDVAS that is to avoid over-correcting of the data.

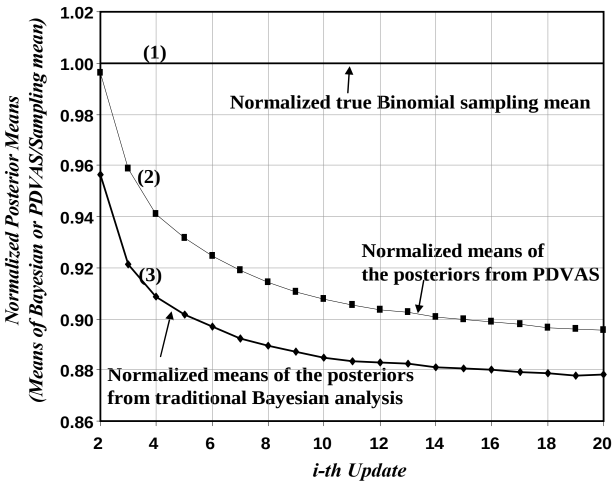

Figure 12. Mean Comparisons of PDVAS and Traditional Bayesian for Example 2

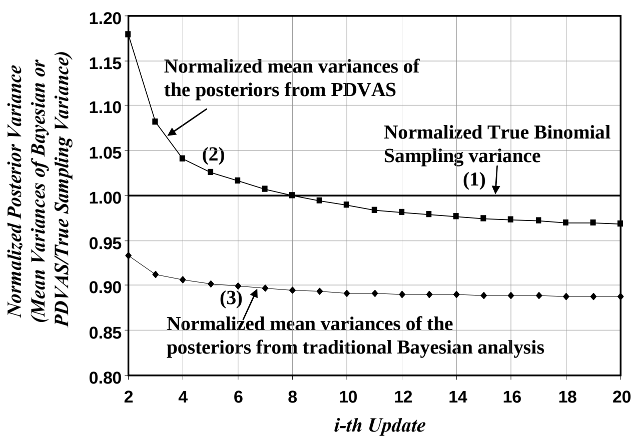

Figure 13. Variance Comparisons of PDVAS and Traditional Bayesian for Example 2

Example 3. All simulation set up in this example is the same as in Example 2 except the difference in 10% data inconsistency anomaly. Instead of inserting 10% of sampling data with Beta(3,97), we insert 10% of sampling data with Beta(30, 70). Mean comparison in Figure 14 shows again the superiority of PDVAS over the tradition Bayesian. Variance comparison in Figure 15 shows PDVAS is worse (bigger variances which are further away from the true sampling variance than the traditional Bayesian's). A detailed examination of the simulation data indicates that the PDVAS algorithm defined in Fig. 3 and 4 does not differentiate the bad data from good well, therefore sometimes throwing away good data that consequently increases the variances. A further research is needed to improve the algorithm to correct this problem.

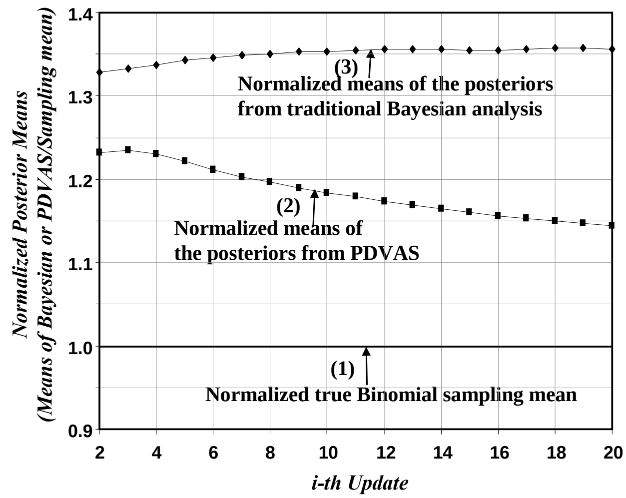

Figure 14. Mean Comparisons of PDVAS and Traditional Bayesian for Example 3

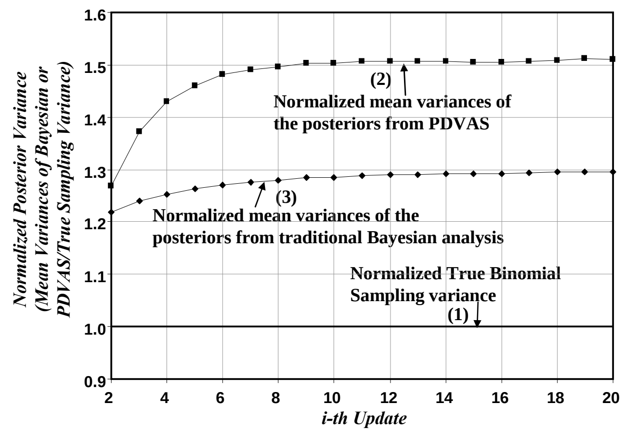

Figure 15. Variance Comparisons of PDVAS and Traditional Bayesian for Example 3

Example 4. This example is to show PDVAS is update sequence dependent and to provide some ideas how much PDVAS posterior means and variances can vary from different update sequences. The property of the update sequence independence is possessed as a nice feature by the traditional Bayesian process with conjugate priors. It is lost in PDVAS traded with data inconsistency check and data adjustment with the intention to produce more valid posterior. In this example, we use the non-informative prior *Beta(0.5, 0.5)* and assume the following 10 data sets are available for Bayesian updating: (2,12), (3,47),(3,38), (7,45), (2,9),(6,57), (1,6), (2,13), (2,10) and (1,99). The first 9 data sets were randomly drawn from the Binomial sampling with $p = 0.10$. The last data set represents a data inconsistency source which is from *Beta(1,99)*. We randomly shuffled the 10 data sets 100 times and applied PDVAS to each of these 100 shuffles. Figure 16 shows the 100 PDVAS posterior means. They are all closer to the sampling mean than the traditional Bayesian which has a fixed mean due to update sequence independence. Similar phenomenon is observed in Figure 17 for the simulated variances from the randomly shuffled data. The noticeable scatter of the means and variances in PDVAS due to update sequence variations brings an open research question that whether it is necessary or possible to further define the PDVAS algorithm to produce an optimized but unique posterior with some pre-defined optimization criteria.

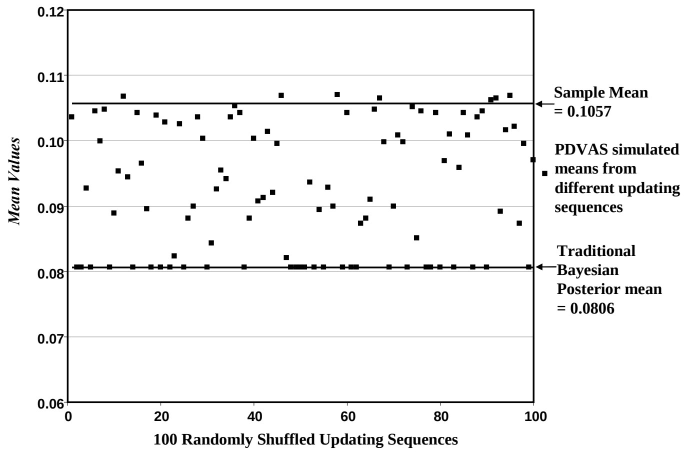

Figure 16. Means of PDVAS Posteriors from 100 Different Update Sequences

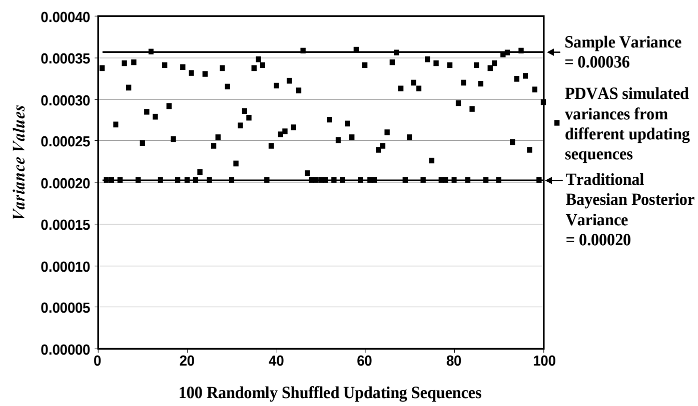

Figure 17. Variances of PDVAS Posteriors from 100 Different Update Sequences

Example 5. This example is to apply PDVAS to a rocket engine reliability analysis. Table 2 presents the data that are assumed to be obtained chronologically as the design maturity evolves. The Bayesian update is performed repeatedly as the new data sets become available to support on-going design decisions.

For the SCA and PBMS combined data set, we interpret as 1 failure in 1,000 trials so (F, S) = (1, 999). If we apply the traditional Bayesian analysis to the above data with a non-informative prior Beta(0.5, 0.5) as the initial prior, we get the final posterior

Beta(0.5+1+1+2+3,0.5+69+13+999+4+18+147+120) ≡ Beta(7.5, 1370.5). This posterior has mean of 7.5/(7.5+1370.5)=0.0054, standard deviation of

$\sqrt{7.5\text{x}1370.5/[(7.5+1370.5)^2(7.5+1370.5+1)]}$ = 0.0020, and COV (coefficient variation, standard deviation/mean) =0.0020/0.0054 = 37%. Now we apply PDVAS to the same data. Table 3 summarizes the traditional Bayesian and PDVAS results in the chronologically manner.

The final traditional Bayesian posterior has mean of 7.5/(7.5+1370.5)=0.0054, standard deviation of $\sqrt{7.5\text{x}1370.5\ /[(7.5+1370.5)^2(7.5+1370.5+1)]}$ = 0.0020 and COV = 0.0020/0.0054 = 37%. The final PDVAS posterior has mean of 2.29/(2.29+1264.42)=0.0018, standard deviation of $\sqrt{2.29\text{x}1264.42\ /[(2.29+1264.42)^2(2.29+1264.42+1)]}$ =0.0012 and COV = 0.0012/0.0018 = 67%. Comparing with the traditional Bayesian, PDVAS posterior has a smaller mean but bigger COV. The mean reduction is because PDVAS discounts several inconsistent data sets which have significant failure probability. The COV increase is because PDVAS discounts the total data sample size (notice for $Beta(\alpha,\beta), COV = \dfrac{\sqrt{\beta}}{\sqrt{\alpha}\sqrt{\alpha+\beta+1}}$). All these are in line with engineering design and development practice that addresses failures when they occur so failure probability is reduced but at the same time, it may introduce new uncertainty and unknowns through redesigns or risk mitigation. PDVAS helps address these concerns by its data validation and adjustment algorithm.

6. Summary and Concluding Remarks

A Bayesian prior and data validation and adjustment scheme (PDVAS) was developed for Beta-Binomial Bayesian model to address the two controversial issues often surrounding the Bayesian reliability analysis, which are the reasonableness of the prior and data consistency. Several properties of PDVAS were presented that provide insights about PDVAS data adjustment strategy, PDVAS posterior's convergence, and update sequence dependency. The PDVAS adjustment formulas were related to reliability data categories from engineering design, and a detailed data adjustment selection framework was provided to assist PDVAS implementation. The PDVAS illustrative examples were given that show the adequateness, effectiveness and usefulness of PDVAS. With PDVAS, Bayesian Reliability Analysis will be more valid with less data inconsistency and contradiction to better support engineering design decision.

There are further research opportunities for PDVAS. Some details of the PDVAS algorithm need to be further refined to accommodate more versatile data situations, and to ensure its convergence. Update sequence dependency of PDVAS seems not avoidable but the optimization of PDVAS posteriors is worthwhile to be explored. The PDVAS data categorization can be more closely defined with engineering design and analysis data as inputs. The generalization of PDVAS can be potentially extended to other prior-data sampling model situations from the special case of Beta-Binomial model as defined in this paper. Finally, PDVAS may provide another criterion to assess the adequacy of the initial prior assignment which has been an active research area for years in Bayesian analysis.

Table 2. Rocket Engine Reliability Data Sets in Example 5

Design stage	Data Category	Number of failures	Number of successes
Concept exploration	Demonstrated reliability from heritage engine A	0	69
	Demonstrated reliability from heritage engine B	0	13
Conceptual Design	Combination of Similarity and Comparative Assessment (SCA) and Physics Based Model and Simulation (PBMS)	Predicted 1 failure per 1000 engine hot fires	
Embodiment Design	Lab test result	1	4
Development	Sub-scale development test results	2	18
	Full scale development test results	3	147
Certification	Certification test results	0	120

Table 3. Traditional Bayesian and PDVAS Comparisons in Example 5

Design stage	Data Category	Number of failures	Number of successes	Traditional Bayesian Posterior	PDVAS Posterior
Initial Prior	Non-informative Beta(0.5,0.5)	0.5	0.5		
Concept exploration	Demonstrated reliability from heritage engine A	0	69		
	Demonstrated reliability from heritage engine B	0	13	Beta(0.5, 82.5)	Beta(0.5, 82.5)
Conceptual Design	Combination of Similarity and Comparative Assessment (SCA) and Physics Based Model and Simulation (PBMS)	1	999	Beta(1.5, 1081.5)	Beta(1.5, 1081.5)
Embodiment Design	Lab test result	1	4	Beta(2.5, 1085.5)	Beta(1.0, 1081.0)
Development	Sub-scale development test results	2	18	Beta(4.5, 1103.5)	Beta(1.01, 1081.09)
	Full scale development test results	3	147	Beta(7.5, 1250.5)	Beta(2.29, 1144.42)
Certification	Certification test results	0	120	Beta(7.5, 1370.5)	Beta(2.29, 1264.42)

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Appendix A. The proofs of the propositions

Proposition 1. For the data set (F_v, S_v) being adjusted in Eq. (13), PDVAS does not change the mean of the data set but increases its standard deviation σ.

Proof: The mean of the data set (F_v, S_v) is $\frac{F_v}{F_v+S_v}$. The mean of the adjusted data set $(\tilde{F}_v, \tilde{S}_v)$ is

$$\frac{\tilde{F}_v}{\tilde{F}_v+\tilde{S}_v} \equiv \frac{S_{DA}F_v}{S_{DA}F_v+S_{DA}S_v} = \frac{F_v}{F_v+S_v}$$

per Eq. (13). For the σ (standard deviation) of $(\tilde{F}_v, \tilde{S}_v)$, denoted as $\tilde{\sigma}_{(\tilde{F}_v\tilde{S}_v)}$, we use σ formula of Beta distribution.

$$\tilde{\sigma}_{Beta(\tilde{F}_v,\tilde{S}_v)} = \frac{\sqrt{\tilde{F}_v\tilde{S}_v}}{(\tilde{F}_v+\tilde{S}_v)\sqrt{\tilde{F}_v+\tilde{S}_v+1}}$$

$$= \frac{\sqrt{S_{DA}F_vS_{DA}S_v}}{(S_{DA}F_v+S_{DA}S_v)\sqrt{S_{DA}F_v+S_{DA}S_v+1}}$$

$$= \frac{\sqrt{F_vS_v}}{(F_v+S_v)\sqrt{S_{DA}(F_v+S_v)+1}} > \frac{\sqrt{F_vS_v}}{(F_v+S_v)\sqrt{(F_v+S_v)+1}}$$

$\equiv \sigma_{(F_v,S_v)} \equiv$ standard deviation of the unadjusted data set (F_v, S_v) for any $S_{DA} < 1$.

Proposition 2. At the i-th Bayesian updating with the data sets (F_0,S_0), (F_1,S_1), ..,, (F_i,S_i), If

$$\frac{F_v}{F_v+S_v} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$

, where $0 \le v \le i$, we then have

$$\frac{F_v}{F_v+S_v} < \frac{F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{F_v+S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} \le \frac{\tilde{F}_v + \sum_{j=0,i\neq v}^{j=i} F_j}{\tilde{F}_v+\tilde{S}_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$

$$\equiv \frac{S_{DA}F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$

Proof: First, we show

$$\frac{F_v}{F_v+S_v} < \frac{F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{F_v+S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}. \quad \text{(A.1)}$$

From $\frac{F_v}{F_v+S_v} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$, we have

$$F_v\left(\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j\right) < (F_v+S_v)\sum_{j=0,i\neq v}^{j=i} F_j$$

Adding $F_v(F_v+S_v)$ on both sides, we get

$F_v(F_v+S_v) + F_v\left(\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j\right) < F_v(F_v+S_v) + (F_v+S_v)\sum_{j=0,i\neq v}^{j=i} F_j$. So we have $F_v\left(F_v+S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j\right) < (F_v+S_v)\left(F_v + \sum_{j=0,i\neq v}^{j=i} F_j\right)$.

Therefore, we have

$$\frac{F_v}{F_v+S_v} < \frac{F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{F_v+S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}$$

. Similarly, we can show the right side inequality of (A.1). For proving

$$\frac{F_v}{F_v+S_v} < \frac{S_{DA}F_v + \sum_{j=0,i\neq v}^{j=i} F_j}{S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j} < \frac{\sum_{j=0,i\neq v}^{j=i} F_j}{\sum_{j=0,i\neq v}^{j=i} F_j + \sum_{j=0,i\neq v}^{j=i} S_j}, \quad \text{(A.2)}$$

we have $\frac{S_{DA}F_v}{S_{DA}F_v+S_{DA}S_v} \equiv \frac{F_v}{F_v+S_v} < \frac{\sum_{j=0,i\neq v}^{j=i}F_j}{\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j}$.

So similar to the proof of (A.1), we have

$$\frac{F_v}{F_v+S_v} \equiv \frac{S_{DA}F_v}{S_{DA}F_v+S_{DA}S_v} < \frac{S_{DA}F_v+\sum_{j=0,i\neq v}^{j=i}F_j}{S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j} < \frac{\sum_{j=0,i\neq v}^{j=i}F_j}{\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j}.$$

Lastly, to prove

$$\frac{F_v+\sum_{j=0,i\neq v}^{j=i}F_j}{F_v+S_v+\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j} \le \frac{S_{DA}F_v+\sum_{j=0,i\neq v}^{j=i}F_j}{S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j}, \quad \text{(A.3)}$$

equivalently, we prove

$$(F_v+\sum_{j=0,i\neq v}^{j=i}F_j)(S_{DA}F_v+S_{DA}S_v+\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j)-$$
$$(F_v+S_v+\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j)(S_{DA}F_v+\sum_{j=0,i\neq v}^{j=i}F_j)\le 0. \quad \text{(A.4)}$$

Reorganizing the left hand side (LHS) of (A.4), we get

$$\text{LHS} = (S_{DA}-1)[(F_v+S_v)\sum_{j=0,i\neq v}^{j=i}F_j - F_v(\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j)]\cdot$$

Since $\frac{F_v}{F_v+S_v} < \frac{\sum_{j=0,i\neq v}^{j=i}F_j}{\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j}$,

$$(F_v+S_v)\sum_{j=0,i\neq v}^{j=i}F_j - F_v(\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j) > 0.$$

Since $S_{DA}\le 1$, $S_{DA}-1\le 0$. Therefore,

$$(S_{DA}-1)((F_v+S_v)\sum_{j=0,i\neq v}^{j=i}F_j - F_v(\sum_{j=0,i\neq v}^{j=i}F_j+\sum_{j=0,i\neq v}^{j=i}S_j))\le 0\,\cdot$$

This proved (A.3).

Proposition 3. There always exists a set of S_{DA} values such that Step 2 and 3 of Figure 4 will converge.

Proof: At the *i-th* Bayesian update, the consistency statistic, calculated by Eq. (10), is given by

$$\chi_C^2 = [\sum_{j=0}^{j=i}(F_j+S_j)]\left(\frac{\sum_{j=0}^{j=i}\frac{F_j^2}{F_j+S_j}}{\sum_{j=0}^{j=i}F_j}+\frac{\sum_{j=0}^{j=i}\frac{S_j^2}{F_j+S_j}}{\sum_{j=0}^{j=i}S_j}-1\right)$$

This is a χ^2 statistic with the degree of freedom of *i*. Under the extreme case, we can have all data sets adjusted with the same data adjustment score S_{DA}. Then the adjusted χ_C^2 value, named as $\tilde{\chi}_C^2$, becomes

$$\tilde{\chi}_C^2 = [\sum_{j=0}^{j=i}(S_{DA}F_j+S_{DA}S_j)]\left(\frac{\sum_{j=0}^{j=i}\frac{(S_{DA}F_j)^2}{S_{DA}F_j+S_{DA}S_j}}{\sum_{j=0}^{j=i}S_{DA}F_j}+\frac{\sum_{j=0}^{j=i}\frac{(S_{DA}S_j)^2}{S_{DA}F_j+S_{DA}S_j}}{\sum_{j=0}^{j=i}S_{DA}S_j}-1\right)$$

$$= S_{DA}[\sum_{j=0}^{j=i}(F_j+S_j)]\left(\frac{\sum_{j=0}^{j=i}\frac{F_j^2}{F_j+S_j}}{\sum_{j=0}^{j=i}F_j}+\frac{\sum_{j=0}^{j=i}\frac{S_j^2}{F_j+S_j}}{\sum_{j=0}^{j=i}S_j}-1\right).$$

Therefore, we can always pick an S_{DA} that is small enough such that $C = P(\chi^2 > \tilde{\chi}_C^2)\ge T_c$.

Proposition 4. For no failure situation, that is, in *(F0,S0)*, *(F1,S1)*, ..,, *(FkSk)*, all F_i=0 except F_0, χ_C^2 by Eq. (10), after substituting F_0 by $\tilde{F}_0 = S_{DA}F_0$ and S_0 by $\tilde{S}_0 = S_{DA}S_0$, is a monotone increase function of S_{DA}, $\chi_C^2 \ge \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k}S_i$ and $\chi_C^2 \to \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k}S_i$ when $S_{DA}\to 0$.

Proof: After substituting F_0 by $\tilde{F}_0 = S_{DA}F_0$, S_0 by $\tilde{S}_0 = S_{DA}S_0$, and F_i by 0 for $i \ge 1$, Eq. (10) becomes

$$\chi_C^2 = [S_{DA}F_0+S_{DA}S_0+\sum_{i=1}^{i=k}S_i]\left(\frac{F_0\sum_{i=1}^{i=k}S_i}{(F_0+S_0)(S_{DA}S_0+\sum_{i=1}^{i=k}S_i)}\right)$$

$$= \frac{F_0+S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}}}{S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}}}\frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k}S_i.$$

It can easily be proven that $(F_0+S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}})/(S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}})$ is a monotone increase function of S_{DA} from calculus. It is also easily seen that when

$S_{DA}\to 0$, $(F_0+S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}})/(S_0+\frac{\sum_{i=1}^{i=k}S_i}{S_{DA}})\to 1$, which leads to $\chi_C^2 \to \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k}S_i$. Since χ_C^2 is a monotone increase function of S_{DA}, therefore $\chi_C^2 \ge \frac{F_0}{F_0+S_0}\sum_{i=1}^{i=k}S_i$.