Validation and Adjustment of Prior and Data for Bayesian Reliability Analysis in Engineering Design

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Bayesian reliability analysis (BRA) technique has been actively used in reliability assessment for engineered systems. However, there are two key controversies surrounding the BRA: the reasonableness of the prior and the consistency among all data sets. These issues have been debated in Bayesian analysis for many years. As we observed, they have not been resolved satisfactorily. These controversies have seriously hindered the applications of BRA as a useful reliability analysis tool to support engineering design. In this paper, a Bayesian reliability analysis methodology with a prior and data validation and adjustment scheme (PDVAS) is developed to address these issues. As the part of the PDVAS development, a consistency measure is first defined that judges the level of consistency among all data sets including the prior. The consistency measure is then used to adjust either the prior or the data or both to the extent that the prior and the data are statistically consistent. This prior and data validation and adjustment scheme is developed for Binomial sampling with Beta prior, called Beta-Binomial Bayesian model. The properties of the scheme are presented. Several illustrative examples are presented, which show the reasonableness, effectiveness, and usefulness of PDVAS. General discussion of the scheme is offered to enhance the Bayesian reliability analysis in engineering design for reliability assessment. [DOI: 10.1115/1.4003841]

Keywords: reliability, Bayesian reliability analysis (BRA), prior distribution, posterior distribution, prior and data validation and adjustment scheme (PDVAS)

1 Introduction

Bayesian reliability analysis (BRA) technique has been actively used in reliability assessment [1–4]. BRA follows a traditional Bayesian approach, which assumes a sampling distribution for the data being analyzed, assigns a probability distribution, called prior distribution, to the sampling distribution parameters, then collects test data to form the likelihood function, and updates the prior to the posterior distribution with the Bayes’ formula that aggregates this new prior with new likelihood data to arrive at an updated posterior distribution. The posterior distribution is then used to form likelihood function in step 3 and derives the posterior distribution in step 4 using the Bayesian formula given by Eq. (1). If a repeated Bayesian updating is conducted, the posterior distribution derived from step 4 loops back in step 4R as a new prior input to step 3, and the Bayesian formula [Eq. (1)] is used again to aggregate this new prior with new likelihood data to arrive at an updated posterior distribution. The posterior distribution is then used as a statistical inference tool in engineering applications as indicated in step 5.

\[
f_{\Theta|X}(\theta|x) = \frac{f_{X}(x|\theta)}{f_X(x)} = \frac{f_{X}(x|\theta)f_{\Theta}(\theta)}{f_X(x)} \quad (1)
\]

In Eq. (1), \(f_{\Theta}(\theta)\) is the prior density for the sampling distribution parameter \(\Theta\), \(f_{X}(x|\theta)\) is the likelihood function of the data \(x\) given a \(\Theta\) value as \(\theta\), \(f_{X}(x|\theta)\) is the posterior density of the \(\Theta\), and \(f_X(x)\) is the marginal density of the data.

If a prior distribution takes the same form of the distribution function as the posterior distribution, the prior is called conjugate prior. Mathematically, the prior, \(f_{\Theta}(\theta)\) in a conjugate Bayesian model, has the same function form as the posterior, \(f_{\Theta|X}(\theta|x)\). The
examples of conjugate prior include Beta prior with Binomial sampling, called Beta-Binomial Bayesian model, and normal prior for the mean with normal sampling. For more conjugate prior discussions, see Refs. [3,4]. A conjugate Bayesian model has several nice features. First, the derivation of the posterior is much easier. Usually, a closed form formula exists that aggregates the prior and the data to obtain the posterior. For example, for the Beta-Binomial model with Binomial parameter \( p \), if we assume \( p \) is subject to \( \text{Beta}(x, \beta) \), and we obtain test data of \( F \) failures out of \( N \) trials, the posterior distribution of \( p \) is then given by \( \text{Beta}(x + F, \beta + N - F) \). Second, for a repeated Bayesian updating with \( k \) (>1) data sets available, the final Bayesian updated posterior is independent of the sequence of the updating taken on the \( k \) data sets.

During an engineering design, especially at the conceptual design stage, reliability analysis relies on various data sources, including historical failure data from similar parts or systems, expert opinions, engineering modeling and simulations, and prototype or laboratory test results. Huang and Jin [14] provided a comprehensive survey of reliability prediction data sources as potential data inputs to BRA. Grantham Lough et al. [15,16] correlated the historical failure data to the functions of system being studied during the functional design to assist risk assessment. Wang and Jin [17] developed a functional design approach, which utilized a Bayesian network technique with uniform distributions as inputs to assess an individual function’s influence on the system failure probability. Wang et al. [1] applied a repeated Bayesian updating technique to a reliability assessment during a product design and development cycle using evolving, insufficient, and subjective data sets, which included a customer survey, response surface physics model results, and clinic trial test data. Huang and Jin [18] extended the traditional reliability stress–strength interference theory to the conceptual design with the combination of a team survey, historical similar function design data, physics bounds of the design to define conceptual stress, and conceptual strength for reliability quantification. However, none of the above work has addressed the data inconsistency and data contradiction issue. For some BRA applications, an obvious data inconsistency may appear between the prior and the test data or among data sets from various data sources. One extreme example is that the prior states that the Binomial sampling parameter \( p \) is equal to 0.01 with probability 1, but the experiment data show five failures out of ten trials. It is obvious that the data set “five out of ten” is very unlikely to come from Binomial sampling with \( p = 0.01 \). A less extreme example is that the prior states that the Binomial parameter \( p \) is subject to the Beta(1,100) prior while the experiment data shows one failure out of ten trials. For data samples such as this, what Bayesian analysis produces is a weighted average of the prior and the data as the posterior, as illustrated in Fig. 2, though the likelihood data and the prior hardly overlap as shown in the figure. A data inconsistency example is when one data set has 1 failure and 2 successes and the other has 1 failure and 50 successes. It is not likely that these two data sets are from the same sampling distribution. Therefore, it is seriously questionable whether these two data sets are combinable for the Bayesian analysis for the inference purpose. The aforementioned prior generation methods, namely maximum entropy method, reference prior analysis, and frequentist matching, only address the generation of the initial prior. As we surveyed Bayesian analysis literature, we observed that there is little active research that addresses the prior and data inconsistency, and the validation of the prior and the data during a repeated Bayesian updating process. The research in Ref. [19] is probably one of the few that discuss the data conflict.

The above discussion brings an obvious need for a prior and data validation and adjustment. If the prior is contradicting with the data, the analyst has three choices: (1) accept the prior and the data and perform Bayesian update as usual; (2) reject the prior and the data; or (3) do something about the prior and the data to continue the Bayesian process in a reasonable manner. For choice 1, the analyst may lead himself or herself to a misleading inference therefore inadequate design decision generated from the Bayesian method. For choice 2, the analyst will have no data to perform Bayesian analysis. For choice 3, the literature survey indicates there is no existing theory and method for doing so. The objective of this paper is to provide such a method to help address choice 3. In Sec. 2, we present a modified Bayesian updating process with an added step that validates the consistency among all data set and adjusts the prior and the data accordingly if inconsistencies arise. We provide the mathematical formulas for the prior and data validation and adjustment scheme (PDVAS) for the Beta-Binomial Bayesian model. In Sec. 3, we present and discuss several properties of PDVAS, which provide some insights of PDVAS. In Sec. 4, we discuss the selection of prior and data adjustment formulas based on the engineering knowledge and data available. In Sec. 5, we present various examples to illustrate the PDVAS applications. We then summarize the paper in Sec. 6, discuss the limitations of PDVAS, and future research possibilities.

2 A PDVAS

2.1 Motivation. As we have pointed out, the traditional Bayesian analysis has some shortcomings in dealing with data contradiction and inconsistency. We believe that one of the key applications of the Bayesian posterior is for inference modeling that predicts the sampling distribution behavior to assist design decisions. Therefore, it is very important for this inference model to be valid. Fundamentally, our motivation is how we can go beyond the simple data aggregation as the Bayesian procedure defines, evaluate data trusting and worthiness, and adjust the data when evidence of inconsistency arises with the engineering design knowledge at hand. The adjustment also needs to balance retaining the knowledge in the data and reducing the data inconsistency. This leads to the proposal of the modified Bayesian analysis flow.

2.2 Modified Bayesian Analysis Flow. Figure 3 presents our proposal of the modified Bayesian analysis process with an added prior and data validation and adjustment step, which is step 4a. All other steps are the same as the original Bayesian analysis flow as in
In Eq. (3), \( \rho \) sets: 0, 1, 2, ..., \( N \), \( p \) is assigned a Beta distribution with parameters \( x \) and \( \beta \). Its density is

\[
p(X = x | \theta = p) = \frac{p^{x-1}(1-p)^{\beta-1}}{B(x, \beta)}
\]

(2)

\( x = 0, 1, 2, ..., N \).

2.3 Prior and Data Validation and Adjustment Formulas. For step 4a of Fig. 3 for the Beta-Binomial model, we have a Binomial distribution as the sampling distribution with the parameter \( p \) as follows

\[
p(X = x | \theta = p) = \binom{N}{x} p^x (1-p)^{N-x} (N-1)
\]

(3)

\( x = 0, 1, 2, ..., N \), \( p \) is assigned a Beta distribution with parameters \( x \) and \( \beta \). Its density is

\[
f(p) = \frac{p^{x-1}(1-p)^{\beta-1}}{B(x, \beta)}
\]

(4)

In Eq. (3), \( 0 < p < 1 \), \( x > 0 \) and \( \beta > 0 \). Beta\((x, \beta)\) is a complete Beta function given by \( \int_0^1 p^{x-1}(1-p)^{\beta-1} dp \). The mean and the standard deviation of the Binomial sampling distribution are

\[
\mu_{\text{Binomial}} = Np
\]

(5)

\[
\sigma_{\text{Binomial}} = \sqrt{Np(1-p)}
\]

(6)

respectively.

The mean and the standard deviation of the Beta\((x, \beta)\) are

\[
\mu_{\text{Beta}} = \frac{x}{x+\beta}
\]

(7)

and

\[
\sigma_{\text{Beta}} = \frac{\sqrt{\beta}}{(x+\beta)\sqrt{x+\beta+1}}
\]

(8)

respectively.

For a general Bayesian analysis, we have the following data:

Prior distribution is given by Beta\((x, \beta)\), and \( k \) data sets (\( k \geq 1 \)) are given by \((F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\). For the convenience of the notation, we name \( F_0 \equiv x \) and \( S_0 \equiv \beta \). So we have Bayesian data sets: \((F_0, S_0), (F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\). Recall \( F_i \) and \( S_i \) represent number of failures and number of successes in the \( i \)th data set, respectively. With the traditional Bayesian process, we obtain the Beta posterior, named Beta\((x', \beta')\). Using Eq. (1), we get

\[
x' = \sum_{i=0}^{i=k} F_i
\]

and

\[
\beta' = \sum_{i=0}^{i=k} S_i
\]

(9)

Remember that \( x' \) and \( \beta' \) are calculated without evaluating the data inconsistency and contradiction among \((F_0, S_0), (F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\). Now the question is how we assess the data consistency and provide a measure of it. We define a consistency statistic with a probability associated with a \( \chi^2 \) statistic as follows

\[
\chi^2 = \left( \sum_{i=0}^{i=k} \left( \frac{F_i^2}{S_i} + \frac{S_i^2}{F_i} \right) - 1 \right)
\]

(10)

The \( \chi^2 \) formula is originated from the \( \chi^2 \) statistic of traditional hypothesis testing for \( 2 \times (k+1) \) contingency table [20,21]. \( \chi^2 \) has a degree of freedom \( k \). Therefore, the consistency for the data sets \((F_0, S_0), (F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\) is defined as

\[
C \equiv \text{Consistency} = P(\chi^2 > \chi^2_C)
\]

(11)

where \( \chi^2_C \) in Eq. (10) is a \( \chi^2 \) random variable with a degree of freedom \( k \).

The motivation of the consistency formulas defined by Eqs. (10) and (11) is that \( F_0 + S_0 = (x + \beta) \) of the prior, per the Bayesian process, represents the prior’s sample size, embedded in the prior knowledge in processing the posterior distribution [4]. \( F_0 \) \((x \approx x)\) and \( S_0 \) \((\beta \approx \beta)\) approximately represent the number of failures and successes, respectively, afforded by the prior. The mean of the prior is \( F_0/(F_0 + S_0) \). If the data are consistent, all the data means, namely \( F_1/(F_1 + S_1), F_2/(F_2 + S_2), ..., F_k/(F_k + S_k) \) should not be very far away from \( F_0/(F_0 + S_0) \). As matter of fact, the data sets of \((F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\), according to a Bayesian assumption, all should be generated from the Binomial sampling with the parameter \( p \), which is subject to a Beta prior with mean \( F_0/(F_0 + S_0) \). Therefore, when the failure fraction, \( F_i/(F_i + S_i), i = 1, 2, ..., k \), of the data sets are equal to or close to the failure fraction of the prior, \( F_0/(F_0 + S_0) \), we, to the maximum extent, believe that the data are consistent, and there is no contradiction between the prior and the data sets. Conversely, if one or more of the \( F_i/(F_i + S_i) \) are drastically different from \( F_0/(F_0 + S_0) \), or some \( F_i/(F_i + S_i) \) is drastically different from \( F_j/(F_j + S_j) \) \((i \neq j)\), we have a reason to think that the data are not consistent and the assumption that all data sets \((F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\) are generated from the same Binomial sampling with \( p \) as the parameter is not adequate. Mathematically, \( F_0/(F_0 + S_0) = F_1/(F_1 + S_1) = \cdots = F_k/(F_k + S_k) \), Eq. (10) leads to \( \chi^2 = 0 \). Therefore, the consistency measure \( C \) by Eq. (11) = 1. When some \( F_i/(F_i + S_i) \) is very much different from some \( F_j/(F_j + S_j) \) \((i \neq j), \chi^2 \) in Eq. (10) becomes very big. Therefore, the consistency measure \( C \) by Eq. (11) is nearly zero.

The consistency measure \( C \), calculated by Eqs. (10) and (11), represents the prior and data validation result. \( C \) is a value between 0 and 1. When \( C = 1 \) or close to 1, we believe that the data sets, \((F_0, S_0), (F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\), are completely consistent or nearly consistent. Thereby we will implement the Bayesian updating unconditionally as we do in the traditional Bayesian process. When \( C = 0 \) or close to 0, we believe that the data sets, \((F_0, S_0), (F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\), are completely inconsistent or nearly inconsistent. Thereby we will seriously challenge the assumption that all data sets, \((F_1, S_1), (F_2, S_2), ..., (F_k, S_k)\), are generated from the same Binomial sampling with \( p \) as the parameter.

Now the question is what do we do when the consistency measure \( C \) by Eq. (11) is between 0 and 1? We propose the following data adjustment algorithm as a part of the modified Bayesian updating process. This is the substantiation of step 4a of the
The exact form of $f$ function in Eq. (12) will be determined in Sec. 4. The $S_{DA}$ also takes values in the range of $[0, 1]$. We then apply the $S_{DA}$ to the data set $(F_v, S_v)$ in the following discounting manner to obtain the discounted values of $F_v$ and $S_v$, namely $\tilde{F}_v$ and $\tilde{S}_v$, respectively.

$$\tilde{F}_v = S_{DA}F_v \quad \text{and} \quad \tilde{S}_v = S_{DA}S_v \quad (13)$$

So far, we have not defined the detailed $f$ functional forms of $S_{DA}$ yet. But we know $S_{DA}$ should satisfy the following conditions

$$S_{DA}(0) = f(0) = 0 \quad (14)$$

and

$$S_{DA}(1) = f(1) = 1 \quad (15)$$

and

$$S_{DA}(\text{consistency}_2) \geq S_{DA}(\text{consistency}_1) \quad \text{when} \quad \text{Consistency}_2 > \text{Consistency}_1 \text{ (monotone increase)} \quad (16)$$

Equations (13)–(16) basically state that when the consistency measure is zero, we completely ignore the data set $(F_v, S_v)$. When the consistency measure is 1, we completely accept the data set $(F_v, S_v)$, which is what the traditional Bayesian updating process does. The bigger the consistency measure, the less we discount the data. Recall that step 1 of Fig. 4 asks for defining a threshold criterion in hypothesis testing is a step function, that is, when $p$-value is below the significance level, the null hypotheses will be rejected. In the context of PDVAS applications, we extend this step function to a set of smooth curves, which can incorporate engineering design and analysis knowledge for data adjustment. In this section, we present general forms of $S_{DA}$ curves first. In Sec. 4, we will provide recommendations as to how a specific curve can be selected based on available engineering knowledge and data.

Figure 6 depicts various possibilities of $S_{DA}$ curves with $T_v = 0.05$. For all the curves in the figure, $S_{DA} = 1$ when the consistency measure $\geq 0.05$. The curve A in the figure is very close to taking all $S_{DA}$ values of 1 for any consistency measure. So if adopted, it defaults to the traditional Bayesian updating process.

The curve G takes almost all $S_{DA}$ values of 0 for the consistency process presented in Fig. 3. Figure 4 shows the algorithm and Fig. 5 is the $\chi^2$ statistic by Eq. (10) using data sets $(F_0, S_0)$, $(F_1, S_1)$, ..., $(F_m, S_m)$, therefore, there is a convergence issue. We will discuss this in Sec. 3. Step 3.3 is to adjust the data set $(F_v, S_v)$, when it is found that it is the source of the inconsistency.

Next we discuss the data adjustment formula. We first define a data adjustment as a mapping from the consistency measure to the data adjustment score, denoted as $S_{DA}$.

$$S_{DA} = f(\text{Consistency}) \quad (12)$$

Fig. 6 Potential candidate functions for $S_{DA}$
measures $\leq 0.05$, which leads to the nearly complete rejection of the data set $(F, S)$, similar to the situation of a traditional hypothesis testing with a significance level of 5% of rejecting a null hypothesis. Section 4 will discuss the details how to select a $T_c$ and $S_{DA}$ curve.

3 Properties of PDVAS

**Proposition 1.** For the data set $(F, S)$ being adjusted in Eq. (13), PDVAS does not change the mean of the data set, but increases its standard deviation $\sigma$.

The proof of Proposition 1 is given in Appendix. Proposition 1 reflects an important PDVAS strategy, that is, to maintain the mean of the adjusted data set but increase the spread of the data distribution. In other words, PDVAS takes the face value embedded in the data regarding to the knowledge of central tendency but discounts the value of the data set by increasing its variance. The rationale of doing so is that when the prior or the data are collected, usually the uncertainty associated with the variability is much bigger than the one associated with the central tendency. Therefore, the variability of the data is more doubtful than the mean. PDVAS focuses on the adjustment of the variability to achieve the data consistency.

**Proposition 2.** At the $i$th Bayesian updating with the $i$ sets of data $(F_0, S_0), (F_1, S_1), \ldots, (F_i, S_i)$ available, if $|\sum_{j=0}^{i} F_j / (\sum_{j=0}^{i} F_j + \sum_{j=0}^{i} S_j)| < 0.05$, the data adjustment does not make $S_{DA}$ value and its function is essentially important. If $S_{DA}$ is too small (close to 0), we tend to disregard good data, which defeats the original intention of the Bayesian analysis. If $S_{DA}$ is too big (close to 1), we accept the data blindly, which may ignore possible data inconsistency and lead to a misleading inference. In this section, we first provide a family of $S_{DA}$ functions and then recommend some criteria for selecting a specific one for general engineering applications.

4 Selection of Data Adjustment Formulas

In Sec. 2, we discussed some general forms of the formulas being applied to data adjustment when PDVAS detects a data inconsistency. Proposition 3 in Sec. 3 indicates that we can always find a data adjustment solution to make the data consistent. However, the intention of PDVAS is to discount the data as less as possible, especially when not being allowed to throw out good data. Therefore, the selection of $S_{DA}$ value and its function is essentially important. If $S_{DA}$ is too small (close to 0), we tend to disregard good data, which defeats the original intention of the Bayesian analysis. If $S_{DA}$ is too big (close to 1), we accept the data blindly, which may ignore possible data inconsistency and lead to a misleading inference. In this section, we first provide a family of $S_{DA}$ functions and then recommend some criteria for selecting a specific one for general engineering applications.

4.1 A Family of $S_{DA}$ Curves

We select the incomplete Beta function within the range $[0, T_c]$ to represent the $S_{DA}$ functions. Here $T_c$ is the threshold value defined in step 1 of Fig. 4. When the consistency measure $C$, calculated by Eq. (11), meets $C \geq T_c$, we accept all the data as is without any adjustment. When the $C$ values fall within $(0, T_c)$, we calculate $S_{DA}$ using an incomplete Beta function as follows:

$$S_{DA} = f(C) = \begin{cases} \int_0^{C/T_c} \left( \frac{u}{T_c} \right)^{m-1} (1-u)^{n-1} du & \text{if } 0 \leq C \leq T_c \\ 1 & \text{if } T_c \leq C \end{cases} \tag{18}$$

(Beta$(m,n)$) in Eq. (18) is a complete Beta function given by $\int_0^1 u^{m-1}(1-u)^{n-1} du$. The rationale of selecting the incomplete Beta function for $S_{DA}$ is its versatility in its shapes and the easy interpretation of the parameters $m$ and $n$ in Eq. (18) related to the engineering knowledge. To illustrate the merit of the Beta function, we present an example of a set of Beta curves in Fig. 7 related to the engineering knowledge.
derivative = 0) near \( m/(m+n)0.05 = 0.0125 \) on the x-axis. As \( m \) and \( n \) get bigger, the curves get steeper around the saddle point, approaching the situation either PDVAS completely rejects the data or completely accepts the data depending on whether \( C \) is bigger than or smaller than 0.0125. It is recognized from a probability theory that the consistency measure \( C \) falls within (0, 0.05), it uniformly takes a value between 0 and 0.05. The family of the curves in Fig. 7 basically state that there is on the average one out of four or 25% (= 0.0125/0.05) chance to significantly reject the data. At the extreme case that we are certainly one out of four times that the data are bad, we completely reject the data when \( C < 0.0125 \). However, in reality we make the statement “one out of four” with uncertainty, so we only reject the data partially, which is quantified by the \( S_{DA} \) values on various curves. Now the question is how we select \( T_{c}, m, \) and \( n \) for the \( S_{DA} \) function for a PDVAS implementation?

### 4.2 Selection of \( T_{c}, m, \) and \( n \)

#### 4.2.1 \( T_{c} \) Selection. \( T_{c} \) represents a screening criterion, which is conceptually similar to the significance level or \( \alpha \) value in a traditional statistical hypothesis testing. The \( \alpha \) value is always set to be small (<0.1) to avoid unacceptable false positive in the hypothesis testing. From an engineering design point view, when we use Bayesian analysis to aggregate the data to predict reliability, data often come from various sources; therefore, big uncertainty can be expected. While we want data inconsistency to be detected, we do not want to go through the data adjustment step when the evidence of data inconsistency is not strong. Therefore, we recommend using \( T_{c} \) value of 0.05 as a standard value for the inconsistency screening. For the case that we want to closely mimic the traditional Bayesian analysis without concerning too much about data inconsistency, we can use a value of 0.01 or smaller. The \( T_{c} \) value of zero defaults PDVAS to the traditional Bayesian process. Thereby, we consider the traditional Bayesian process is just a special case of PDVAS.

#### 4.2.2 \( m,n \) Selection. As we mentioned in Sec. 4.1, the ratio of \( m/(m+n) \) approximately represents our knowledge and judgment based on engineering knowledge on the possible percentage of inconsistent data. While keeping \( m/(m+n) \) unchanged, the more we can pinpoint the source of data inconsistency with certainty, the bigger the \( m \) and \( n \) we can assign to. As one of our early research papers surveyed [14], reliability data for a Bayesian analysis can come from the following four sources: (1) statistical frequency method (SF); (2) similarity and comparative assessment (SCA); (3) physics based modeling and simulation (PBMS); and (4) expert elicitation (EE). Usually, SF data, which are generated from field or laboratory simulated operating environment, have the smallest modeling uncertainty for the reason that fitted statistical inference model partially addresses the sampling modeling uncertainty. Uncertainty in other three data sources (SCA, PBMS, and EE) can vary widely but data from EE probably have the biggest modeling uncertainty since they are purely based on expert judgment. With the above assessment, we use Table 1 as the selection frame work for \( m \) and \( n \), which also serves as an illustrative example.

Column (1) in Table 1 classifies the data source categories. We use our research result in Ref. [14] to divide all possible data sources into four categories (SF, SCA, PBMS, and EE). Users of PDVAS can create their own data source categories. Column (2) is the engineering judgment of the analyst based on their knowledge on the design and analysis for what the average percentage of the inconsistent data can be. Column (3) is to assess the uncertainty of column (2), which asks approximately how many times the analyst has experienced the data inconsistency instances in the past. By the Beta function definition represented by Eq. (18), the number of inconsistency data instances equates to the parameter \( n \) value. So column (4) = column (3), \( n \) in Eq. (18) represents the estimated number of good data instances the analyst experienced in the past. So, \( n \) value in column (5) is back-calculated using Eq. (19). Column (6) completely defines the \( S_{DA} \) Beta function used in Eq. (18). Figure 8 creates the \( S_{DA} \) Beta curves based on the data from Table 1.

\[
\frac{m}{m+n} = \text{column(2)}
\]  

Figure 8 indicates that we rarely reject SF data in a dramatic manner shown in curve (1). This is in line with our recognition that SF data often is the most reliable data source. Comparing curve (2) [Beta(2,2)] and curve 3 [Beta(1,1)], both of them partially discount the data in a prorated fashion. Curve (3) is strictly linear. Curve (2) discounts the data less when \( C > 0.025 \) and discounts more when \( C < 0.025 \) than curve (3) does. This is because we are more confident on Beta(2,2) than on Beta(1,1), which, by derivation, indicates we have experienced more bad data instances in Beta(2,2) situation than in Beta(1,1). For curve (4) in the figure, we discount the majority of the data (more than 50%) when \( C \) falls below 0.035 (70% of 0.05).

### Table 1 \( m,n \) selection framework with an example

<table>
<thead>
<tr>
<th>Datacategory</th>
<th>Estimated percentage of inconsistent data (%)</th>
<th>Number of analyst’s experiences on bad data</th>
<th>( S_{DA} ) Beta function in Eq. (18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF</td>
<td>5</td>
<td>1</td>
<td>Beta(1,19)</td>
</tr>
<tr>
<td>SCA</td>
<td>50</td>
<td>2</td>
<td>Beta(2,2)</td>
</tr>
<tr>
<td>PBMS</td>
<td>50</td>
<td>1</td>
<td>Beta(1,1)</td>
</tr>
<tr>
<td>EE</td>
<td>70</td>
<td>3</td>
<td>Beta(3,1.29)</td>
</tr>
</tbody>
</table>
It is recognized that the data categories are created with subjectivity, the above approach is only considered to be a framework. Users of PDVAS can create their own data categories and assign $m$ and $n$ with their own knowledge and judgment. We recommend, for a quick analysis or for the situations that not much information is available about the data sources, the user use the linear curve (Beta(1,1)) as the $S_{DA}$ function as Fig. 9 shows, which basically discounts the data in a linearly prorated manner when $C/C_2 < 0.05$ without assessing the fraction of possible bad data. In many of our PDVAS simulation runs, this approach is proven to be reasonable.

5 Case Examples

In this section, various examples are presented to illustrate PDVAS applications. The results from examples 1 to 4 are obtained through Monte Carlo simulations that assume certain Bayesian prior and data sampling distributions. Examples 1–3 show the effectiveness of PDVAS that detects data inconsistencies and adjusts the data accordingly. Example 4 shows the updating sequence dependency of PDVAS. Example 5 is to apply PDVAS to a rocket engine reliability analysis with various data sources.

Example 1. We assume the initial prior distribution is Beta(1,99), but the sampling distributions all come from Binomial with $p = 0.10$. Intuitively, there is a data inconsistency between the prior and the data sets because the mean of prior is $1/(1 + 99) = 0.01$ while sampling mean is $p = 0.10$. We run 20 Bayesian updates following the process outlined in Fig. 3. The results are obtained through a computer simulation using the PDVAS algorithm presented in Figs. 4 and 5. We use $T_c = 0.05$ and the linear curve for $S_{DA}$ as in Fig. 9 to execute PDVAS. Figure 10 presents the mean comparison. Line (1) in the figure is the true sampling mean (normalized to $p = 0.1$). Curve (2) is the Monte Carlo simulated average means from 10,000 Monte Carlo runs using PDVAS. Curve (3) is the average means of traditional Bayesian updated posteriors. This example illustrates that the PDVAS is effective in correcting the data for the case that an initial prior is too optimistic. The figure also indicates that all three curves tend to converge together eventually when $i$ (number of updates) goes to infinity. However, PDVAS mean is much closer to the true sampling mean for the small number of updates. Therefore, PDVAS is very useful for the practical situations with small number of Bayesian updates conducted. Figure 11 presents the variance comparisons, which also indicates that PDVAS predicts the posterior variance much closer to the true sampling variance than the traditional Bayesian process does. This is because the traditional Bayesian blindly takes the initial prior Beta(1,99) as a part of the posterior updating, which significantly increases total sample size in the final posterior distribution, therefore underestimates the variance.
Example 2. Similar to example 1, we assume the sampling distribution is a Binomial with \( p = 0.10 \). Initial prior is the noninformative prior Beta(0.5, 0.5). However, in this example, we insert a data inconsistency anomaly by randomly generating 10% of the data sampling with Beta(3, 97) as the sampling distribution. Notice Beta(3, 97) has a much smaller mean \[ \frac{3}{3+97} = 0.03 \] than the Binomial sampling mean \( p = 0.10 \). We want to test, in a simulated practical application, whether PDVAS can detect the data inconsistency as a multiple Bayesian updating is executed to incorporate newly obtained data. Again we use \( T_c = 0.05 \) and the linear \( S_{DA} \) curve. Figures 12 and 13 present the mean and variance of PDVAS posteriors from 100 different update sequences.
comparisons, respectively. The results from the figures show that PDVAS is superior to the tradition Bayesian since both means and variances of PDVAS posteriors are closer to the true sampling mean and variance than the tradition Bayesian posteriors after three updates. It is interesting to notice that both PDVAS and traditional Bayesian means and variances diverge from the true sampling mean and variance. This is because there is always 10% of the sampling data with Beta(3,97) as their distribution with a smaller mean and variance than the sampling mean and variance. PDVAS for this case, even though better than the traditional Bayesian process, is not detecting all data inconsistency, which is intentional in the design of PDVAS that is to avoid overcorrecting of the data.

**Example 3.** All simulation set up in this example is the same as in example 2 except the difference in 10% data inconsistency anomaly. Instead of inserting 10% of sampling data with Beta(3,97), we insert 10% of sampling data with Beta(30,70). Mean comparison in Fig. 14 shows again the superiority of PDVAS over the tradition Bayesian. Variance comparison in Fig. 15 shows PDVAS is worse (bigger variances which are further away from the true sampling variance than the tradition Bayesian variances). A detailed examination of the simulation data indicates that the PDVAS algorithm defined in Figs. 3 and 4 does not differentiate the bad data from the good well, increasing the variance due to some of the good data being thrown out. Further research is needed to improve the algorithm to deal with this problem, which will be discussed in Sec. 6.

**Example 4.** This example is to show that PDVAS is update sequence dependent and to provide some ideas for how much PDVAS posterior means and variances can vary from different update sequences. The property of the update sequence independence is possessed as a nice feature by the traditional Bayesian process with conjugate priors. It is lost in PDVAS traded with the data inconsistency check and data adjustment with the intention to produce more valid posterior. In this example, we use the noninformative prior Beta(0.5,0.5) and assume the following ten data sets are available for Bayesian updating: (2,12), (3,47),(3,38), (7,45), (2,9),(6,57), (1,6), (2,13), (2,10), and (1,99). The first nine data sets are randomly drawn from the Binomial sampling with $p = 0.10$. The last data set represents a data inconsistency source, which is from Beta(1,99). We randomly shuffled the 10 data sets 100 times and applied PDVAS to each of these 100 shuffles. Figure 16 shows that all 100 PDVAS posterior means (symbolized by dark squares) are closer to the sampling mean (the upper line) than to the traditional Bayesian posterior mean (the lower line), which is a fixed value due to the update sequence independence. Similar phenomenon is observed in Fig. 17 for the simulated variances. The noticeable scatter of the means and variances in PDVAS due to update sequence variations brings an open research question whether it is necessary or possible to further define the PDVAS algorithm to produce an optimized but unique posterior with some predefined optimization criteria. We will discuss this in Sec. 6.

**Example 5.** This example is to apply PDVAS to a rocket engine reliability analysis. Table 2 presents the data that are assumed to be chronologically obtained as the design maturity evolves. The Bayesian update is performed repeatedly as the new data sets be chronologically obtained as the design maturity evolves. The property of the update sequence independence is possessed as a nice feature by the traditional Bayesian process with conjugate priors. It is lost in PDVAS traded with the data inconsistency check and data adjustment with the intention to produce more valid posterior. In this example, we use the noninformative prior Beta(0.5,0.5) and assume the following ten data sets are available for Bayesian updating: (2,12), (3,47),(3,38), (7,45), (2,9),(6,57), (1,6), (2,13), (2,10), and (1,99). The first nine data sets are randomly drawn from the Binomial sampling with $p = 0.10$. The last data set represents a data inconsistency source, which is from Beta(1,99). We randomly shuffled the 10 data sets 100 times and applied PDVAS to each of these 100 shuffles. Figure 16 shows that all 100 PDVAS posterior means (symbolized by dark squares) are closer to the sampling mean (the upper line) than to the traditional Bayesian posterior mean (the lower line), which is a fixed value due to the update sequence independence. Similar phenomenon is observed in Fig. 17 for the simulated variances. The noticeable scatter of the means and variances in PDVAS due to update sequence variations brings an open research question whether it is necessary or possible to further define the PDVAS algorithm to produce an optimized but unique posterior with some predefined optimization criteria. We will discuss this in Sec. 6.

**Table 2** Rocket engine reliability data sets in example 5

<table>
<thead>
<tr>
<th>Design stage</th>
<th>Data category</th>
<th>Number of failures</th>
<th>Number of successes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept exploration</td>
<td>Demonstrated reliability from heritage engine A</td>
<td>0</td>
<td>69</td>
</tr>
<tr>
<td>Conceptual design</td>
<td>Demonstrated reliability from heritage engine B</td>
<td>0</td>
<td>13</td>
</tr>
<tr>
<td>Embodiment design</td>
<td>Combination of SCA and PBMS</td>
<td>0</td>
<td>13</td>
</tr>
<tr>
<td>Development</td>
<td>Laboratory test result</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Certification</td>
<td>Subscale development test results</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>Certification</td>
<td>Full scale development test results</td>
<td>3</td>
<td>147</td>
</tr>
<tr>
<td>Certification</td>
<td>Certification test results</td>
<td>0</td>
<td>120</td>
</tr>
</tbody>
</table>

**Table 3** Traditional Bayesian and PDVAS comparisons in example 5

<table>
<thead>
<tr>
<th>Design stage</th>
<th>Data category</th>
<th>Number of failures</th>
<th>Number of successes</th>
<th>Traditional Bayesian posterior</th>
<th>PDVAS posterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial prior</td>
<td>Noninformative</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Concept exploration</td>
<td>Demonstrated reliability from heritage engine A</td>
<td>0</td>
<td>69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conceptual design</td>
<td>Demonstrated reliability from heritage engine B</td>
<td>0</td>
<td>13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Embodiment design</td>
<td>Combination of SCA and PBMS</td>
<td>1</td>
<td>999</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Development</td>
<td>Laboratory test result</td>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Development</td>
<td>Subscale development test results</td>
<td>2</td>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Development</td>
<td>Full scale development test results</td>
<td>3</td>
<td>147</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Certification</td>
<td>Certification test results</td>
<td>0</td>
<td>120</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For the SCA and PBMS combined data set, we interpret as 1 failure in 1000 trials so \( (F_S) = (1.999) \). If we apply the traditional Bayesian analysis to the above data with a noninformative prior \( \text{Beta}(0.5, 0.5) \) as the initial prior, we get the final posterior \( \text{Beta}(0.5 + 1 + 2 + 3.05 + 69 + 13 + 999 + 4 + 18 + 147 + 120) \) or \( \text{Beta}(7.5, 1370.5) \). This posterior has a mean of 7.5/(7.5 + 1370.5) = 0.0054, a standard deviation of \( \sqrt{7.5 \times 1370.5} \times \sqrt{(7.5 + 1370.5)^2} \) = 0.0020, and a COV of 0.0020/0.0054 = 37%. Now we apply PDVAS to the same data. Table 3 summarizes the traditional Bayesian and PDVAS results in the chronologically manner. The final PDVAS posterior has a mean of 2.29/(2.29 + 1264.42) = 0.0018, a standard deviation of \( \sqrt{2.29 \times 1264.42} \times \sqrt{(2.29 + 1264.42)^2} \) = 0.0012 and a COV of 0.0012/0.0018 = 67%. Comparing with the traditional Bayesian, PDVAS posterior has a smaller mean (0.0018 versus 0.0020) but bigger COV (67% versus 37%). The mean reduction is because PDVAS discounts several inconsistent data sets, which have significant failure probability. The COV increase is because PDVAS discounts the total data sample size (notice for Beta(\( \alpha, \beta \)), COV = \( \sqrt{\frac{\beta}{(\sqrt{\frac{1}{\alpha} + \beta + 1})} } \)). All these results are in line with the engineering design and development practice that addresses failures when they occur so the failure probability is reduced but at the same time, redesign of the product or implementation of some corrective actions may introduce new uncertainty and unknowns, which leads to a bigger COV.

6 Summary and Concluding Remarks

A Bayesian PDVAS was developed for the Beta-Binomial Bayesian model to address the two controversial issues surrounding the Bayesian reliability analysis, which are the reasonableness of the prior and data consistency. PDVAS attempts to balance retaining the knowledge in the data and reducing the data inconsistency. PDVAS is also devised so that the traditional Bayesian becomes a special case of it and is a default position if we do not have enough knowledge to adjust the data. Several properties of PDVAS were presented that provide insights about PDVAS data adjustment strategy, PDVAS posterior’s convergence, and update sequence dependency.

There are some limitations for PDVAS. One limitation is that PDVAS aims at detecting and reducing data noise and disturbance that provide insights about PDVAS data adjustment strategy, PDVAS posterior’s convergence, and update sequence dependency. The mean of the PDVAS derived posterior in that example is a default position if we do not have enough knowledge to adjust the data. Several properties of PDVAS were presented that provide insights about PDVAS data adjustment strategy, PDVAS posterior’s convergence, and update sequence dependency. These limitations are as follows:

- **Lack of generalization**: PDVAS is not for data comprising that leads to overestimating the sampling variance. The results of these two cases indicate that a proper balance of the risk of rejecting good data and the risk of failing to detect data inconsistency may not be easily achieved. As discussed early, PDVAS is an update sequence dependent; therefore, PDVAS applications on conjugated prior data sets need to be cautioned for possible noticeably different posteriors due to different update sequences.

- **Sensitivity to prior and data consistency**: PDVAS attempts to balance retaining the mean of the PDVAS derived posterior in that example is a default position if we do not have enough knowledge to adjust the data. Several properties of PDVAS were presented that provide insights about PDVAS data adjustment strategy, PDVAS posterior’s convergence, and update sequence dependency. These limitations are as follows:

There are further research opportunities for PDVAS. One immediate interest is to extend the PDVAS approach and formulas developed in this paper for the Beta-Binomial model to general prior-sampling distribution situations. Quick examinations of the PDVAS algorithm presented by Figs. 4 and 5 in Sec. 2 and five propositions presented in Sec. 3 indicate that they all can be easily generalized to other prior-sampling distribution situations. The PDVAS consistency measure (Eq. (11) in Sec. 2) and the data adjustment formulas (Eqs. (18) and (19) in Sec. 4) will have to be developed for a specific prior and sampling distribution pair of interest.

Other research opportunities are as follows. Some details of the PDVAS algorithm need to be refined to accommodate more versatile data situations for balancing the two risks (the risk of rejecting good data and the risk of failing to detect data inconsistency). The selection criteria of the PDVAS screen threshold (\( T_f \)) needs to be more rigorously established, which links to a data validity measure and data acceptance and rejection risks. Update sequence dependency of PDVAS is unavoidable but the optimization of PDVAS posteriors is worthwhile to explore that can lead to uniqueness of the posterior. The PDVAS data categorization can be more closely defined with engineering design and analysis data inputs. Finally, PDVAS may provide another criterion to assess the adequacy of the initial prior assignment, which has been an active research area for years in Bayesian analysis.

Appendix: The Proofs of the Propositions

**Proposition 1.** For the data set \((F_i, S_i)\) being adjusted in Eq. (13), PDVAS does not change the mean of the data set but increases its standard deviation \( \sigma \).

**Proof.** The mean of the data set \((F_i, S_i)\) is \( \bar{F} = (F_i + S_i) \). The mean of the adjusted data set \((\tilde{F}_i, \tilde{S}_i)\) is \( \tilde{F} = (\tilde{F}_i + \tilde{S}_i) \), where

\[
\tilde{F} = \sqrt{\frac{F_i S_i}{D_F + D_S}} \left( \frac{F_i + S_i}{F_i + S_i} \right) \frac{1}{D_F + D_S} \frac{1}{F_i + S_i} + 1 > \frac{F_i + S_i}{F_i + S_i} + 1
\]

\[= \frac{F_i + S_i}{F_i + S_i} + 1 \]

\[\equiv \sigma_{(F, S)} \]

is standard deviation of the unadjusted data set \((F_S, S_S)\) for any \( D_F < 1 \).

**Proposition 2.** At the ith Bayesian updating with the i sets of data \((F_{0i}, S_{0i})\), \((F_{1i}, S_{1i})\), ..., \((F_{ni}, S_{ni})\) available, if

\[
\sum_{i=0}^{n} \left( F_i + S_i \right) < \sum_{j=0}^{n} \left( F_j + S_j \right)
\]

where \( 0 \leq n \leq i \), and \((F', S')\) is one of the i data sets, we then have

\[
F_{i} \leq \sum_{j=0}^{n} F_{j} \left( \frac{F_{i} + S_{i}}{F_{i} + S_{i}} \right) \frac{1}{D_F + D_S} \frac{1}{F_i + S_i} + 1
\]

\[= \frac{F_i + S_i}{F_i + S_i} + 1 \]

\[\equiv \sigma_{(F, S)} \]

is standard deviation of the unadjusted data set \((F_S, S_S)\) for any \( D_F < 1 \).

051003-10 / Vol. 133, May 2011 Transactions of the ASME

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Proof. First, we show

$$\frac{F_v}{F_v + S_v} < \frac{\sum_{j=0}^{j=i} F_j}{\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

(A1)

From $[F_v/(F_v + S_v)] < [\sum_{j=0}^{j=i} F_j/\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j]$, we have $[F_v, \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j] < [(F_v + S_v) \sum_{j=0}^{j=i} F_j]$

Adding $F_v(F_v + S_v)$ on both sides, we get

$$F_v(F_v + S_v) < F_v(F_v + S_v) + (F_v + S_v) \sum_{j=0}^{j=i} F_j$$

So we have $[F_v(F_v + S_v) + \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j] < [(F_v + S_v) \sum_{j=0}^{j=i} F_j]$. Therefore, we have

$$\frac{F_v}{F_v + S_v} < \frac{\sum_{j=0}^{j=i} F_j}{F_v + S_v + \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

Similarly, we can show the right side inequality of (A1).

For proving

$$\frac{S_{DA}F_v}{S_{DA}F_v + S_{DA}S_v} < \frac{\sum_{j=0}^{j=i} F_j}{\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

we have

$$\frac{S_{DA}F_v}{S_{DA}F_v + S_{DA}S_v} = \frac{\sum_{j=0}^{j=i} F_j}{\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

So, similar to the proof of (A1), we have

$$\frac{F_v}{F_v + S_v} < \frac{\sum_{j=0}^{j=i} F_j}{\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

Lastly, to prove

$$\frac{F_v + \sum_{j=0}^{j=i} F_j}{F_v + S_v + \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j} \leq \frac{S_{DA}F_v + \sum_{j=0}^{j=i} F_j}{S_{DA}F_v + S_{DA}S_v + \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}$$

(A3)

equivalently, we prove

$$\left( F_v + \sum_{j=0}^{j=i} F_j \right) \left( \frac{S_{DA}F_v + S_{DA}S_v + \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j}{S_{DA}F_v + \sum_{j=0}^{j=i} F_j} \right) \leq 0$$

(A4)

Reorganizing the left-hand side (LHS) of (A4), we get

$$\text{LHS} = \left( S_{DA} - 1 \right) \left( (F_v + S_v) \sum_{j=0}^{j=i} F_j - \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j \right)$$

Since $[F_v/(F_v + S_v)] < [\sum_{j=0}^{j=i} F_j/\sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j]$

$$(F_v + S_v) \sum_{j=0}^{j=i} F_j - \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j > 0$$

Since $S_{DA} \leq 1$, $S_{DA} - 1 \leq 0$. Therefore,

$$\left( S_{DA} - 1 \right) \left( (F_v + S_v) \sum_{j=0}^{j=i} F_j - \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j \right) \leq 0$$

This proved (A3).

Proposition 3. There always exists a set of $S_{DA}$ values such that steps 2 and 3 of Fig. 4 will converge.

Proof. At the $i$th Bayesian update, the consistency statistic, calculated by Eq. (10), is given by

$$\chi_i^2 = \left( \sum_{j=0}^{j=i} (F_j + S_j) \right) \left( \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j - 1 \right)$$

This is a $\chi^2$ statistic with the degree of freedom of $i$. Under the extreme case, we can have all data sets adjusted with the same data adjustment score $S_{DA}$. Then the adjusted $\chi_i^2$ value, named as $\tilde{\chi}_i^2$, becomes

$$\tilde{\chi}_i^2 = \left( \sum_{j=0}^{j=i} (S_{DA}F_j + S_{DA}S_j) \right) \left( \sum_{j=0}^{j=i} S_{DA}F_j + \sum_{j=0}^{j=i} S_{DA}S_j - 1 \right)$$

$$= S_{DA} \left( \sum_{j=0}^{j=i} (F_j + S_j) \right) \left( \sum_{j=0}^{j=i} F_j + \sum_{j=0}^{j=i} S_j - 1 \right)$$
Therefore, we can always pick an $S_{DA}$ that is small enough such that $C = P(\chi^2 > \chi^2_C) \geq T_c$.

**Proposition 4.** For the no-failure situation, that is, in $(F_0, S_0), (F_1, S_1), \ldots, (F_k, S_k)$, all $F_i = 0$ except $F_0$, $\chi^2_C = \sum_{i=1}^{k} S_i \geq \left\{ (F_0/(F_0 + S_0)) \sum_{i=1}^{k} S_i \right\}$ and $\chi^2_C \rightarrow (F_0/(F_0 + S_0)) \sum_{i=1}^{k} S_i$ when $S_{DA} \rightarrow 0$.

**Proof.** After substituting $F_0 \rightarrow S_{DA}F_0, S_0 \rightarrow S_{DA}S_0, and F_i \rightarrow 0$ for $i \geq 1, \text{Eq. (10)}$ becomes

$$
\chi^2_C = S_{DA}F_0 + S_{DA}S_0 + \sum_{i=1}^{k} S_i \left( \frac{F_0 \sum_{i=1}^{k} S_i}{(F_0 + S_0)(S_{DA}S_0 + \sum_{i=1}^{k} S_i)} \right)
$$

$$
= F_0 + S_0 + \sum_{i=1}^{k} S_i \frac{S_{DA}}{F_0 + S_0} \frac{F_0 \sum_{i=1}^{k} S_i}{S_{DA}S_0 + \sum_{i=1}^{k} S_i}
$$

It can easily be proven that $[F_0 + S_0 + (\sum_{i=1}^{k} S_i / S_{DA})] / [S_0 + (\sum_{i=1}^{k} S_i / S_{DA})]$ is a monotone increase function of $S_{DA}$ from calculus. It is also easily seen that when $S_{DA} \rightarrow 0, [F_0 + S_0 + (\sum_{i=1}^{k} S_i / S_{DA})]/[S_0 + (\sum_{i=1}^{k} S_i / S_{DA})] \rightarrow 1$, which leads to $\chi^2_C \rightarrow (F_0/F_0 + S_0) \sum_{i=1}^{k} S_i$. Since $\chi^2_C$ is a monotone increase function of $S_{DA}$, therefore $\chi^2_C \geq (F_0/(F_0 + S_0)) \sum_{i=1}^{k} S_i$.

**References**


